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On irreversible microscopic processes

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Abstract
Irreversible microscopic processes (viz intrinsic stochasticities), i.e. processes in which a unitary dynamical group is related to a semi-group of contracting maps, are considered for the following examples: the shift in $\mathbb{Z}$ w.r.t. random walks; the shift in $\mathbb{R}$ w.r.t. diffusion; the nonrelativistic free Hamiltonian and the relativistic free Hamiltonian of scalar spin-zero particles with rest-mass w.r.t. the corresponding contractive semi-group generated by these operators. The last example is also considered in the context of a particle–antiparticle system, thereby exhibiting an asymmetry between the number of particles and antiparticles. The positive and negative parts of the Hamilton operator of the Dirac equation are calculated and related to an intrinsic stochasticity. For classical Hamiltonian systems an intrinsic stochasticity is defined and applied to examples. Reverse processes and measurements connected with intrinsic stochasticity are defined.

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1. Introduction

The concept of *microscopic irreversibility* (or *intrinsic stochasticity*) had been introduced in a number of papers by Misra, Prigogine and Courbage ([1–4]; see also [16] for a variation of this theme). It had been taken up in the context of a W^*-algebra approach in [5]. Based on this approach we shall present a number of examples whose time evolutions and associated contractive semi-groups have generators which are either functions of momentum operators or classical Hamiltonians. We consider also an example in which a discrete shift could turn into a random process.

To explain the concept of *microscopic irreversibility* let us consider an example. Given a quantum dynamical system with a time-flow $f \to \alpha_t f := \exp(-\mathrm{i}tH) f \exp(\mathrm{i}tH)$, where t denotes the time, H the Hamilton operator of the system (assuming that this operator is strictly positive and has an absolutely continuous spectral part) and f is from a class of operators $f, g, \ldots$ such that $\langle f, g\rangle := \mathrm{trace}(f^*g)$ defines an inner product (that is, $f, g, \ldots$ are Hilbert–Schmidt operators). Define then a map $f \to \tilde{\alpha}_t f := \exp(-tH) f \exp(-tH)$. Note that the α_t, $t \in \mathbb{R}$, form a group of unitary automorphisms, that is, $\langle g, \alpha_t f\rangle = \langle \alpha_{-t} g, f\rangle$ and $\alpha_t(fg) = (\alpha_t f)(\alpha_t g)$. With a restriction $t_0 \leqslant t, t_0 \leqslant 0$, the $\tilde{\alpha}_t$ form a semi-group of strictly contractive maps, that is, $\tilde{\alpha}_t f \to 0$ for $t \to \infty$. The problem we are going to consider is as follows: Does there exist an operator Λ such that

$$(*) \quad \tilde{\alpha}_t \Lambda f = \Lambda \alpha_t f, \qquad t_0 \leqslant t, t_0 \leqslant 0,$$

on a certain subset of elements f?

If this is the case, we shall say that the system has an intrinsic or inner stochasticity, meaning that it allows spontaneous irreversible or dissipative processes according to

$$(**) \quad \|\tilde{\alpha}_t \Lambda f\| \geqslant \|\tilde{\alpha}_{t'} \Lambda f\|, \qquad t \geqslant t'.$$

The operator Λ will occasionally be called a *Lyapunov operator* and the process connected with it a *Lyapunov process* (cf [17]).

The term 'microscopic irreversibility' shall express the spontaneity of these processes and thus distinguish them from dissipative processes on a macroscopic scale which take statistically place in large macroscopic ensembles. Let for example H be the Hamilton operator of a relativistic spin-zero type particle with a rest-mass $m_0 \neq 0$. It will be shown that this system has an intrinsic stochasticity such that for certain f there holds

$$\|\tilde{\alpha}_t f\|^2 = \mathrm{const}\exp(-2m_0 t)[2m_0/t + O(1/t^2)], \qquad t > 0.$$

This suggests to connect this relation with a particle decay (For a nonrelativistic free particle one obtains a completely different result, namely $\|\tilde{\alpha}_t f\|^2 = \mathrm{const}(1+t/2m_0)^{-3/2}$). In another example we shall consider a coupled Boson spin-zero particle–antiparticle system with a rest-mass m_0. It turns out that dynamics (as expressed by the maps α_t) and intrinsic stochasticity (as expressed by the maps $\tilde{\alpha}_t$) are nontrivially together possible only if there is an asymmetry between the number of particles and antiparticles. This might explain why there are more particles than antiparticles (or vice versa, depending on the kind of universe) in connection with an arrow of time.

In another example we have investigated a flow of discrete shifts which could turn into a random motion.

As to spin 1/2 type systems we deliver a preliminary result by constructing for the Dirac equation with a constant electro-magnetic field, whose Hamiltonian H is indefinite, a positive definite operator $|H| \equiv H_+ - H_-$ where H_+ and H_- are the positive and negative parts of the corresponding Hamilton operator H, respectively. This allows us to define the contracting maps by $F \to \tilde{\alpha}_t F = \exp(-t|H|)F\exp(-t|H|)$.

We have further considered classical dynamical systems with a time-flow $\alpha_t = \exp(tL_h)$, where L_h denotes the Liouville operator of the Hamilton function h. In contrast to the quantum mechanical case the maps α_t are 'outer' automorphisms. Contractive maps are therefore not given in a straightforward canonical fashion. Either by working with quantization maps or, equivalently, with a blunt translation from a quantum into a corresponding classical system one arrives at $\tilde{\alpha}_t = \exp(-2th)$ (this reminds strongly of a Gibbs factor in classical statistical mechanics). It is perhaps not surprising that for example for a classical relativistic particle we obtain a result that agrees qualitatively with the above given result in the quantum mechanical case. Since practical calculations for classical systems are considerably easier (at least for integrable systems), we could provide more nontrivial examples for such systems. For nonintegrable Hamiltonians (systems that can become chaotic) calculations are complicated due to the lack of first integrals needed for the integration of the corresponding differential equations whose solutions determine the operators Λ. So far we have not been able to calculate the intrinsic stochasticity for such a system.

A *reverse process* is defined by the relation

$$(***) \quad \alpha_t K f = K\tilde{\alpha}_t f.$$

It will be shown that the operators $K\Lambda$ and ΛK commute with the Hamilton operator of the considered system (note that K does not have to be the inverse of Λ, that is, $K\Lambda$ and ΛK are not necessarily a multiple of the unit operator). This suggests a balance between both processes (one might consider a reverse process also in connection with time reversal).

Intrinsic stochasticity implies by the way via the map Λ the definition of entropy operators (viz *Microscopic Entropy*) which satisfy *inter alia* certain relations in connection with particle number operators. The construction of the state space in which we are going to work has also some interesting features since it requires an ascending chain of state spaces, according to a growing complexity of systems. This has first been observed by Misra [4] who has shown that the operators Λ and $M = \Lambda^*\Lambda$ do not commute with all classical observables (meaning that one needs a state space that is 'larger' than that in which classical mechanics works).

Results which refer to physics are mostly either given as theorems or examples or explained in physical terms. In a few cases results are given by integrals that cannot be explicitly calculated. Lemmata are prooftechnical tools. Corollaries and propositions express in most cases mathematical fineries (which, too, can have a physical meaning). In order not to get lost in what seems predominantly mathematical lingo one can skip these statements and look for theorems and examples which refer to physics.

Although the examples and features demonstrated above have very suggestive physical interpretations they remain so far speculative as regards physics (it is mildly comforting that they share this property with some other theories, like for example string theory).

2. Mathematical preliminaries

2.1. Basics

Part of the mathematical set up we apply here had been first developed in [5]. It is based on an approach that has proved to be more concise and efficient, last not least as regards practical calculations (using extensively for a number of examples formal manipulations by computer algebra). The starting point will be an algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$ of bounded linear operators on some Hilbert space $\mathcal{H}$. By $\bar{\mathcal{A}}$ we shall denote the set of all not necessarily bounded linear operators with domain and range in $\mathcal{H}$. Let τ denote the trace for $\mathcal{H}$. The Hilbert space $\mathbb{H} \equiv L^2(\mathcal{A}, \tau)$, which will then serve as a state space, is constructed as follows (the reason why we need this state space rather than the Hilbert space $\mathcal{H}$ will become clear in what follows after a few technicalities). Let $\mathcal{A}_+$ be the positive cone of $\mathcal{A}$ (i.e. the set of all positive operators in $\mathcal{A}$), and let $\mathcal{A}_1 = \text{span}\{a \in \mathcal{A}_+ | \tau(a) < \infty\}$ and $\mathcal{A}_2 = \text{span}\{a \in \mathcal{A} | \tau(a^*a) < \infty\}$. Then $\mathcal{A}_2$ is a unitary algebra with an inner product $\langle a, b\rangle := \tau(a^*b)$. $\mathcal{A}_1$ and $\mathcal{A}_2$ are two-sided ideals of $\mathcal{A}$ and $\mathcal{A}_2\mathcal{A}_2 = \mathcal{A}_1$. Let (cf [7]) $L^k \equiv L^k(\mathcal{A}, \tau), k \in \{1, 2\}$, denote the completion of $\mathcal{A}_k$ w.r.t. the norm $\|x\|_k := (\tau(|x|^k))^{1/k}$, $|x| := (x^*x)^{1/2}$. In classical mechanics $\mathcal{A} = L^\infty(X, \mu)$, where X is a subset of phase space and $\tau(x)$ is the norm $\|x\|_1$ of $L^1(X, \mu)$. Thus $L^k(\mathcal{A}, \tau) = L^k(X, \mu), k \in \{1, 2, \infty\}$. In quantum mechanics let $\mathcal{A} = \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space of, say, Schrödinger or Fock vectors and $\tau(x) = \text{trace}(x)$. Then L^1 and L^2 are the algebras of trace-class and Hilbert–Schmidt operators, respectively. L^2 is in particular a Hilbert algebra. To connect this with usual formalism let $A \in \mathcal{L}(\mathcal{H})$ be an observable and ρ a density matrix (defining a state on $\mathcal{A}$). Then $0 < \rho \in L^1(\mathcal{A}, \tau), \tau(\rho) = 1$. Now, since

$\langle f, g\rangle := \tau(f^*g)$ is an inner product for $\mathbb{H}$, there is an element $f \in \mathbb{H} = L^2(\mathcal{A}, \tau)$ so that $\langle A\rangle_\rho \equiv \tau(\rho A) = \langle f, Af\rangle$, $\rho = ff^*$, $\tau(\rho) = \langle f, f\rangle = \|f\|_2^2 = 1$.

We shall make use of the following operation. If T is a linear operator with domain and range in $\mathbb{H}$ then $T \to T^\times$ shall be defined by $Tf \to T^\times f := (Tf^*)^*$, $f \in \mathrm{dom}(T)$. A linear operator T with domain and range in $\mathbb{H}$ is said to be *finitely implemented* (by $\bar{\mathcal{A}}$) if there are $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ in $\bar{\mathcal{A}}$ so that $Tf = A_1 f B_1 + \cdots + A_n f B_n$ for all $f \in \mathrm{dom}(T)$. T is said to be *-*invariant* if $T^\times f := (Tf^*)^* = Tf$.

We repeat now the definition given by relation ($*$) above somewhat more detailed.

Let $\alpha = \{\alpha_t | t \in \mathbb{R}\}$ be a unitary group of automorphisms defining a time-flow of a dynamical system. A strongly continuous unitary representation $\alpha : \mathbb{R} \to \mathcal{L}(\mathbb{H})$ is said to have an *intrinsic* or *innerstochasticity* w.r.t. a contractive strongly continuous representation $\tilde{\alpha} : \mathbb{R}_+ \to \mathcal{L}(\mathbb{H})$ if there is a densely defined closed linear operator Λ with domain and range in $\mathbb{H}$ which has a densely defined inverse such that

$$\tilde{\alpha}_t \Lambda = \Lambda \alpha_t \qquad \text{for all} \quad t \in \mathbb{R}_+ \tag{1}$$

on a dense domain D (the density of D is nontrivial because Λ is necessarily unbounded, cf [5]).

In one of the examples below we shall consider a group of automorphisms consisting of shifts $S : f(n) \to (Sf)(n+1)$, $n \in \mathbb{Z}$. To include this in a more general definition of intrinsic stochasticity we generalize the above definition as follows.

Let $\mathcal{G}$ be an ordered locally compact abelian group with a positive cone $\mathcal{G}_+$ (that is, $\mathcal{G}_+$ is a closed semi-subgroup of $\mathcal{G}$, cf [8]; note that this excludes compact groups). A strongly continuous unitary representation $\alpha : \mathcal{G} \to \mathcal{L}(\mathbb{H})$ is said to have an *intrinsic* or *innerstochasticity* w.r.t. a contractive strongly continuous representation $\tilde{\alpha} : \mathcal{G}_+ \to \mathcal{L}(\mathbb{H})$ if there is a densely defined closed linear operator Λ with domain and range in $\mathbb{H}$ which has a densely defined inverse such that

$$\tilde{\alpha}_g \Lambda = \Lambda \alpha_g \qquad \text{for all} \quad g \in \mathcal{G}_+ \tag{2}$$

on a dense domain D.

In the just mentioned example of discrete shifts the group $\mathcal{G}$ is then the additive group $\mathbb{Z}$ of integers and $\mathcal{G}_+$ the set (the semi-subgroup) $\mathbb{Z}_+$ of positive integers. Requiring local compactness insures that there is a dual group $\hat{\mathcal{G}}$ related to $\mathcal{G}$ by Fourier transform, cf [8]. This will be useful for proof techniques. That is, for $\mathcal{G} = \mathbb{R}$ we have $\hat{\mathcal{G}} = \mathbb{R}$ whereas the dual group of $\mathcal{G} = \mathbb{Z}$ is $\hat{\mathcal{G}} = [-\pi, \pi](\mathrm{mod}\, 2\pi)$.

Let $M = \Lambda^*\Lambda$. Then (2) implies for $0 \neq f \in D$ and all $g, g' \in \mathbb{R}_+$, $g \leqslant g'$,

$$\langle \alpha_g f, M\alpha_g f\rangle \geqslant \langle \alpha_{g'} f, M\alpha_{g'} f\rangle > 0 \tag{3}$$

or, equivalently,

$$\|\Lambda\alpha_g f\| = \|\tilde{\alpha}_g \Lambda f\| \geqslant \|\Lambda\alpha_{g'} f\| = \|\tilde{\alpha}_{g'}\Lambda f\| > 0. \tag{4}$$

This shows the dissipativity of the system $\{\alpha, \tilde{\alpha}, \Lambda\}$. (Note that so far Λ is not unique. For let A and B be bounded invertible linear operators which commute with all α_g and all $\tilde{\alpha}_g$, $g \in \mathcal{G}_+$. Then Λ in relation (2) could be replaced $A\Lambda B$.)

In the examples for classical dynamical systems we let $\mathcal{A} = L^\infty(X, \mu)$ where either $X = \mathbb{Z}$ or $X = \mathbb{R}$ and μ is the corresponding Haar measure, that is $\mathbb{H} = L^2(X, \mu)$ (this is essentially the mathematical framework for classical mechanics as established by Koopman more than 75 years ago, cf [6, 7]).

In the quantum mechanical examples we have $\mathcal{A} = \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is either a Schrödinger or a Fock space, and $\mathbb{H} = L^2(\mathcal{A}, \tau)$ is the two-sided ideal of Hilbert–Schmidt operators in $\mathcal{A}$ set up as a Hilbert space by an inner product $\langle f, g\rangle := \tau(f^*g)$. We have shown above that this is equivalent with the usual density matrix calculus (normal states on $\mathcal{A}$). Let $\mathcal{A}^{(1)} \equiv \mathcal{L}(\mathbb{H})$ be the set of bounded and $\bar{\mathcal{A}}^{(1)}$ be the set of not necessarily bounded linear operators with domain and range in $\mathbb{H}$. The crucial fact is (cf [5]) that Λ and M cannot be elements of $\bar{\mathcal{A}}$ but are in $\bar{\mathcal{A}}^{(1)} \backslash \bar{\mathcal{A}}$ (meaning that they do not operate in $\mathcal{H}$). Under certain conditions they cannot even be finitely implemented by elements of $\bar{\mathcal{A}}$. This explains why in quantum mechanics, that is $\mathcal{A} = \mathcal{L}(\mathcal{H})$, one cannot use $\mathcal{H}$ as the relevant Hilbert space in which Λ has domain and range. A convenient realization of the state space is then $\mathbb{H} = \mathcal{H} \otimes \mathcal{H}^*$ where $\mathcal{H} \to \mathcal{H}^*$ means complex conjugation (cf [5]) so that Λ is necessarily affiliated with the full tensor product $\mathcal{L}(\mathcal{H}) \otimes \mathcal{L}(\mathcal{H}^*)$. (For a proof that $\mathcal{L}(\mathbb{H})$ is the strong closure of operators which are finitely implemented by $\mathcal{A}$ see appendix B.)

That Λ and M are not in $\bar{\mathcal{A}}$ has in particular for classical dynamical systems an important consequence because it means that these operators do not commute with all classical observables. This has first been recognized by Misra, cf [4]. For a quantum mechanical system in which dynamics is determined by a group $\alpha : \mathbb{R} \to \mathcal{L}(\mathbb{H}) : t \to \alpha_t = \exp(\mathrm{i}t\delta), \delta f := Hf - H^\times f \equiv Hf - fH$, where the Hamiltonian H is s.a. nonnegative and an element of $\bar{\mathcal{A}}$, there is a canonical way to define an intrinsic stochasticity by letting $\tilde{\alpha}_t \mathbb{R}_+ \to \mathcal{L}(\mathbb{H}) : t \to \tilde{\alpha}_t = \exp(-t\tilde{\delta}), \tilde{\delta} f := Hf + fH$.

Remark 1. The trivial decomposition $H \equiv \tilde{\delta}/2 + \delta/2\mathrm{i}$ allows with regard to its components a physical interpretation: $\tilde{\delta}/2$ can be considered as the proper energy operator (cf [15, 16]) whereas $\delta/(2\mathrm{i})$ refers to frequencies (recall that for stationary systems the frequencies $\omega_{mn} \equiv (E_m - E_n)/\hbar$ are eigenvalues of $\delta/2\mathrm{i}\hbar$).

The examples we shall consider have with the exception of classical Hamiltonians in common that the generators of α and $\tilde{\alpha}$ are functions of momentum operators. This allows basically the same mathematical treatment.

It should be noted that equation (1) requires (cf [5]) α to have a nonvoid continuous spectral part. We shall further require that Λ is *-invariant, that is, $\Lambda^\times f = (\Lambda f^*)^* = \Lambda f$ for all f in the domain of Λ, and that Λ preserves the volume, that is, $\langle \mathbf{1}, \Lambda f\rangle = \langle \mathbf{1}, f\rangle$ for all positive $f \in \mathrm{dom}(\Lambda) \cap L^1(\mathcal{A}, \tau)$; in addition M should be positivity preserving (depending on the specific definition of positivity).

2.2. *Using Fourier transforms*

We shall provide now four lemmata which will be useful for proving statements in the subsequent sections. These lemmata are purely technical and can be skipped by those who are only interested in statements which refer to physics in the sections to follow.

In the following $\mathcal{G}$ will be either $\mathbb{Z}$ or $\mathbb{R}$ so that the dual group $\hat{\mathcal{G}}$ is either $\hat{\mathbb{Z}} = [-\pi, \pi](\mathrm{mod}\, 2\pi)$ or $\hat{\mathbb{R}} = \mathbb{R}$ respectively. Let $X = \mathcal{G}^N, \mathbb{H} = L^2(X)$ (with respect to the Haar measure on X) and $\hat{\mathbb{H}} = L^2(\hat{\mathbb{H}})$. Let further $\mathcal{F} : \mathbb{H} \to \hat{\mathbb{H}}$ denote the Fourier operator, and let $\hat{\Lambda} = \mathcal{F}\Lambda\mathcal{F}^{-1}$, $\beta_g = \mathcal{F}\alpha_g\mathcal{F}^{-1}$ and $\tilde{\beta}_g = \mathcal{F}\tilde{\alpha}_g\mathcal{F}^{-1}$. Then equation (2) is euivalent to

$$\tilde{\beta}_g \hat{\Lambda} = \hat{\Lambda} \beta_g, \qquad g \in \mathcal{G}_+. \tag{5}$$

It will be generally assumed here that

$$\beta_g = \exp(\mathrm{i}g\omega(k)), \qquad \tilde{\beta}_g = \exp(-g\tilde{\omega}(k)), \tag{6}$$

where $k = (k_1 \ldots k_N) \in \hat{X}$ and ω and $\tilde{\omega}$ are quadratic or linear forms.

Lemma 1. *Let $\vartheta : \hat{X} \to \mathbb{C}$ so that $\mathrm{i}\omega(\vartheta(k)) = -\tilde{\omega}(k)$. Le $\mathcal{L}_\vartheta$ be the linear set of all $\mathbb{C}$-valued functions which are analytic in a neighbourhood of each $\vartheta(k)$, $k \in \hat{X}$. For $\hat{\lambda}$, $\hat{f} \in \mathcal{L}_\vartheta$, $\hat{\lambda}$ fixed, and sufficiently small $\varepsilon > 0$ define (writing $(z-\zeta)^{-1} = (z_1-\zeta_1)^{-1}\dots(z_N-\zeta_N)^{-1}$)*

$$(\hat{\Lambda}\hat{f})(k) = (2\pi\mathrm{i})^{-N} \int_{|z-\vartheta(k)|=\varepsilon} \hat{\lambda}(z)\hat{f}(z)(z-\vartheta(k))^{-1}\,\mathrm{d}z, \qquad z \in \mathbb{C}^N. \quad (7)$$

Then (5) holds on $\mathcal{L}_\vartheta$.

Proof. $(\hat{\Lambda}\hat{f})(k) = \hat{\lambda}(\vartheta(k))\hat{f}(\vartheta(k))$ by (6). Hence

$$\begin{aligned}(\tilde{\beta}_g\hat{\lambda}\hat{f})(k) &= \exp(-g\tilde{\omega}(k))\hat{\lambda}(\vartheta(k))\hat{f}(\vartheta(k)) \\ &= \hat{\lambda}(\vartheta(k))\hat{f}(\vartheta(k))\exp(\mathrm{i}g\omega(\vartheta(k))) \\ &= (\hat{\lambda}\beta_g\hat{f})(k).\end{aligned}$$

□

Lemma 2. *Assume that*

$$\int_{\hat{X}} |\hat{\lambda}(\vartheta(k))|^2 \exp(C|\vartheta(k)|)\,\mathrm{d}k < \infty$$

for each positive C. Then Λ is densely defined.

Proof. Let $\hat{X}_1, \hat{X}_2, \dots$ be an ascending chain of measurable compact subsets of $\hat{X}$ so that for each $\hat{h} \in \mathbb{H}$

$$\lim_{n\to\infty} \int_{\hat{X}\setminus\hat{X}_n} |\hat{h}(k)|^2\,\mathrm{d}k = 0.$$

Let $f \in C_c(X)$ (= functions with compact support). Then (cf [8]) $\hat{f} = \mathcal{F}f$ is entire analytic and grows at most exponentially. Hence there exist positive constants C_1 and C_0 and a natural n_0 so that for all $n \geqslant n_0$,

$$\left| \|\Lambda f\|^2 - \int_{\hat{X}_n} |(\hat{\lambda}\hat{f})(\vartheta(k))|^2\,\mathrm{d}k \right| \leqslant C_0 \int_{\hat{X}\setminus\hat{X}_n} |\hat{\lambda}(\vartheta(k))|^2 \exp(C_1|\vartheta(k)|)\,\mathrm{d}k.$$

□

Thus by hypothesis $\|\hat{\Lambda}\hat{f}\| = \|\Lambda f\| < \infty$. Since $C_c(X)$ is dense in $\mathbb{H}$ the assertion follows.

Lemma 3. *Let $\vartheta_* = \vartheta^{-1}$. Define $\hat{\kappa}$ by $\hat{\kappa}(\vartheta_*(k)) = \hat{\lambda}(k)^{-1}$, $k \in \hat{X}$, and assume that*

(a) $\hat{\kappa} \in \mathcal{L}_{\vartheta_*}$ (= set of functions which are analytic in a neighbourhood of each $\vartheta_*(k)$, $k \in \hat{X}$);
(b) $\int_{\hat{X}} |\hat{\kappa}(\vartheta_*(k))|^2 \exp(C|\vartheta_*(k)|\,\mathrm{d}k < \infty$ for each positive constant C.

Let $h \in \mathcal{L}_{\vartheta_*}$ and define for sufficiently small $\varepsilon > 0$

$$(\hat{K}\hat{h})(k) = (2\pi\mathrm{i})^{-N} \int_{|z-\vartheta_*(k))|=\varepsilon} \hat{\kappa}(z)\hat{h}(z)(z-\vartheta_*(k))^{-1}\,\mathrm{d}z, \qquad z \in \mathbb{C}^N. \quad (8)$$

Then $K = \mathcal{F}^{-1}K\mathcal{F}$ is densely defined and the inverse of Λ.

Proof. Copying the proof of lemma 2 the density of the domain of K follows from $C_c(X) \subset \mathrm{dom}(K)$. It suffices therefore to prove that $\hat{K}$ is the inverse of $\hat{\Lambda}$. This follows from

$$(\hat{K}\hat{\Lambda}\hat{f})(k) = \hat{\kappa}(\vartheta_*(k))(\hat{\lambda}\hat{f})(\vartheta(\vartheta_*(k))) = \hat{\lambda}(k)^{-1}\hat{\lambda}(k)\hat{f}(k) = \hat{f}(k).$$

By the boundedness of α_g and $\tilde{\alpha}_g$ it follows from (1):

$$\tilde{\alpha}_g\Lambda^{**} = \Lambda^{**}\alpha_g \qquad \text{for all} \quad g \in \mathcal{G}_+.$$

□

It can be inferred from this

Lemma 4. *If $\tilde{\alpha}_g\Lambda = \Lambda\alpha_g$, $g \in \mathcal{G}_+$, on a dense domain then one can assume Λ to be closed and $M = \Lambda^*\Lambda$ to be essentially selfadjoint positive.*

3. Shift and random walk in $\mathbb{Z}$

Theorem 1. *Let $\mathbb{H} = L^2(\mathbb{Z})$ and let $(Sf)(n) = f(n+1)$, $f \in \mathbb{H}, n \in \mathbb{Z}$. Then the unitary group $\alpha : \mathbb{Z} \to \mathcal{L}(\mathbb{H}) : n \to S^n$ has an intrinsic stochasticity with respect to the strictly contractive semi-group of random walks in $\mathbb{Z}$,*

$$\tilde{\alpha} : \mathbb{Z}_+ \to \mathcal{L}(\mathbb{H}) : n \to (pS + qS^*)^n, 0 < p < 1, 0 < q < 1, p + q = 1.$$

Proof. With $\hat{f}(k) = (\mathcal{F}f)(k) = \sum_{m\in\mathbb{Z}} f(m)\exp(\mathrm{i}mk)$ it follows $(\hat{S}\hat{f}(k) = \exp(-\mathrm{i}k)\hat{f}(k)$, $\hat{S} = \mathcal{F}S\mathcal{F}^{-1}$. Thus relation (1) reads

$$[\cos(.) - \mathrm{i}(p-q)\sin(.)]\hat{\Lambda} = \hat{\Lambda}\exp(-\mathrm{i}(.)). \tag{9}$$

The function ϑ of lemma 1 is therefore here

$$\vartheta(k) = \mathrm{i}\ln[\cos(k) - \mathrm{i}(p-q)\sin(k)]. \tag{10}$$

Choose now

$$\hat{\lambda}(z) = \hat{c}(z)\exp(az^2), \qquad a = \text{const} > 0, \tag{11}$$

where $\hat{c}$ is in $\mathcal{L}_\vartheta$ and grows at most exponentially on the range of ϑ. Let $f \in C_c(X)$. Clearly, $\hat{\lambda} \in \mathcal{L}_\vartheta$ and $(\hat{\Lambda}\hat{f})(\vartheta(k))$ is bounded on $[-\pi, \pi]$. Hence $\|\hat{\Lambda}\hat{f}\| = \|\Lambda f\| < \infty$. Thus Λ is densely defined. In particular

$$(\Lambda f)(n) = (2\pi)^{-1}\sum_{m\in\mathbb{Z}} f(m)\int_{-\pi}^{\pi}\exp(-\mathrm{i}nk)\hat{\lambda}(\vartheta(k))\exp(\mathrm{i}m\vartheta(k))\,\mathrm{d}k. \tag{12}$$

Further,

$$\vartheta_*(k) = \vartheta^{-1}(k) = -\mathrm{i}\ln\{[\exp(-\mathrm{i}k) + \mathrm{i}(4pq - \exp(-2\mathrm{i}k))^{1/2}]/2q\}. \tag{13}$$

Clearly, ϑ_* is bounded on $[-\pi, \pi]$. Let $\hat{\kappa}(\vartheta_*(k)) = \hat{\lambda}(k)^{-1} = \hat{c}(k)^{-1}\exp(-ak^2)$. Assume further that the function $k \to \hat{c}(k)^{-1}$ is analytic in a neighbourhood of $[-\pi, \pi]$ and grows at most exponentially (let p.e. $\hat{c}(z) = C\exp(Cz)$). Then $\hat{K}$ of lemma 3 is a densely defined inverse of Λ. Finally, since $C_c(\mathbb{Z})$ is invariant under S, it follows that

$$(pS + qS^*)^n\Lambda = \Lambda S^n, n \in \mathbb{Z}_+, 0 < p < 1, 0 < q < 1, p + q = 1,$$

holds on $C_c(\mathbb{Z})$. □

The following corollaries refer to theorem 1.

Corollary 1. $\Lambda = \Lambda^\times$ *if* $\overline{\hat{\lambda}(\vartheta(-k))} = \hat{\lambda}(\vartheta(k))$.

Proof. Since $\overline{\vartheta(-k)} = -\vartheta(k)$ it follows from (10)

$$\begin{aligned}\overline{\Lambda\bar{f}(n)} &= (2\pi)^{-1}\sum_{m\in\mathbb{Z}} f(m)\int_{-\pi}^{\pi}\overline{\hat{\lambda}(\vartheta(k))}\exp\{\mathrm{i}[nk - m\overline{\vartheta(k)}]\}\,\mathrm{d}k\\ &= (2\pi)^{-1}\sum_{m\in\mathbb{Z}} f(m)\int_{-\pi}^{\pi}\overline{\hat{\lambda}(\vartheta(-k))}\exp\{\mathrm{i}[-nk + m\vartheta(k)]\}\,\mathrm{d}k.\end{aligned}$$ □

Corollary 2. *M is positivity preserving if $p = q$ and the integral of $|\hat{\lambda}(\vartheta(k))|^2\cos^{-m-n}(k)$ on $[-\pi, \pi]$ is nonnegative.*

Proof. By the relation (δ = Dirac distribution)

$$(2\pi)^{-1}\sum_{n\in\mathbb{Z}}\exp(\mathrm{i}kn) = \sum_{n\in\mathbb{Z}}\delta(k - 2\pi mn) \tag{14}$$

and standard regulation it follows with (8) and (10) for $p = q$,

$$(Mf)(n) = (2\pi)^{-1} \sum_{m\in\mathbb{Z}} f(m) \int_{-\pi}^{\pi} |\hat{\lambda}(\vartheta(k))|^2 \cos^{-m-n}(k)\, \mathrm{d}k. \tag{15}$$

□

By (12) and regularization it follows also

Corollary 3. $\hat{c}(0)\langle \mathbf{1}, \Lambda f\rangle = \langle \mathbf{1}, f\rangle$ *for all* $f \in L^1(\mathbb{Z})$.

Note that $\hat{c}(z)$, which was defined in relation (11), can always be chosen such that $\hat{c}(0) = 1$.

4. Shift and diffusion in $L^2(\mathbb{R})$

Let now $\mathcal{G} = \mathbb{R}, \mathcal{A} = L^\infty(\mathbb{R})$ and $\mathbb{H} = L^2(\mathbb{R})$. Let further $P = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$ and $\hat{f}(k) \equiv (\mathcal{F}f)(k) = (2\pi)^{-1/2}\int_{\mathbb{R}} \exp(-\mathrm{i}kx) f(x)\, \mathrm{d}x$.

Theorem 2. *The group of shifts,* $\alpha : \mathbb{R} \to \mathcal{L}(\mathbb{H}) : t \to \alpha_t = \exp(\mathrm{i}tP)$*, has an intrinsic stochasticity w.r.t. the strictly contractive semi-group* $\tilde{\alpha} : \mathbb{R}_+ \to \mathcal{L}(\mathbb{H}) : t \to \tilde{\alpha}_t = \exp(-tP^2)$.

Proof. It follows from $(\mathcal{F}P\mathcal{F}^{-1}\hat{f})(k) = k\hat{f}(k)$ that $\vartheta(k) = -\mathrm{i}k^2, k \in \mathbb{R}$. Choose now

$$\hat{\lambda}(z) = \hat{c}(z)\exp(az^2), \qquad a = \text{const} > 0, \tag{16}$$

where $\hat{c}$ is in $\mathcal{L}_\vartheta$ and grows at most exponentially on the range of ϑ. Let $f \in C_c^\infty(\mathbb{R})$. Then (cf [8]) $\hat{f}$ is entire analytic and grows at most exponentially on the range of ϑ. By (1) $\hat{f}$ is in $\mathrm{dom}(\hat{\Lambda})$, hence $C_c^\infty(\mathbb{R}) \subset \mathrm{dom}(\Lambda)$ and therefore Λ is densely defined. In particular

$$\begin{aligned}(\Lambda f)(x) &= (2\pi)^{-1/2}\int_{\mathbb{R}} \exp(\mathrm{i}kx)\hat{\lambda}(-\mathrm{i}k^2)\hat{f}(-\mathrm{i}k^2)\, \mathrm{d}k \\ &= (2\pi)^{-1/2}\int_{\mathbb{R}} \left[\exp(-ak^4 + \mathrm{i}kx)\hat{c}(-\mathrm{i}k^2)\int_{\mathbb{R}} \exp(-yk^2) f(y)\, \mathrm{d}y\right] \mathrm{d}k \\ &= (2\pi)^{-1/2}\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \hat{c}(-\mathrm{i}k^2)\exp(-ak^4 - yk^2 + \mathrm{i}xk)\, \mathrm{d}k\right] f(y)\, \mathrm{d}y.\end{aligned} \tag{17}$$

□

Assume in addition that $\hat{c}^{-1}$ is analytic and grows at most exponentially. Then κ in lemma 3 has the required properties and hence determines a densely defined inverse K of Λ. Finally, since $C_c^\infty(\mathbb{R})$ is invariant under α_t for all $t \in \mathbb{R}$, it follows that relation (1) holds on the dense domain $C_c^\infty(\mathbb{R})$.

Corollary 4. $\Lambda = \Lambda^\times$ *if* $\overline{\hat{c}(-\mathrm{i}k^2)} = \hat{c}(-\mathrm{i}k^2)$.

Corollary 5. $M = \Lambda^*\Lambda$ *preserves positivity.*

Proof. It follows from (15) by regularization

$$(Mf)(x) = (2\pi)^{-1/2}\int_{\mathbb{R}} \left\{\int_{\mathbb{R}} |\hat{c}(-\mathrm{i}k^2)| \exp[-2ak^4 - (x+y)k^2]\, \mathrm{d}k\right\} f(y)\, \mathrm{d}y.$$

Clearly, M has a positive integral kernel. This proves the assertion. □

Also by regularization one proves

Corollary 6. $\hat{c}(0)\langle \mathbf{1}, \Lambda f\rangle = \langle \mathbf{1}, f\rangle$ *for all* $f \in L^1(\mathbb{R})$.

(Again we can here and in the examples to follow always choose $\hat{c}$ such that $\hat{c}(0) = 1$ so that volume preserving is guaranteed.)

In the following $\mathcal{E}'(\mathbb{R}^N)$ denotes the linear space of distributions with compact support in N variables.

Corollary 7. *For arbitrary $u \in \mathcal{E}'(\mathbb{R})$ there holds*

(a) $\|\tilde{\alpha}_t \Lambda u\| = \|\Lambda \alpha_t u\| < \infty, t \geqslant 0$*;*
(b) $0 < -\mathrm{d}\|\Lambda \alpha_t u\|/\mathrm{d}t < \infty, t \geqslant 0$*;*
(c) $\hat{c}(0)\langle \mathbf{1}, \Lambda u\rangle = \langle \mathbf{1}, u\rangle$ *if u is in addition integrable.*

Proof. $\mathcal{E}'(\mathbb{R})$ is invariant under $\alpha_t, t \in \mathbb{R}$, and the Fourier–Laplace transform of an element $u \in \mathcal{E}'(\mathbb{R})$ is (cf [8]) an entire analytic function that grows at most exponentially. Thus $\Lambda u = \mathcal{F}^{-1}\hat{\Lambda}\hat{u} \in \mathbb{H}$ by the properties of $\hat{\Lambda}$. This proves (a); (b) follows from the strong continuity of $\tilde{\alpha}$ and α and the strict contractiveness of $\tilde{\alpha}$. By regularization one obtains (c). □

Remark 2. By the preceding corollary dom(Λ) contains deterministic states, that is, elements $u_\varepsilon \in \mathrm{dom}(\Lambda)$ with the properties $\mathrm{supp}(u_\varepsilon) \subset [x - \varepsilon, x + \varepsilon], x \in \mathbb{R}, \varepsilon$ arbitrary > 0 and $\|u_\varepsilon\| = 1$ for all $\varepsilon > 0$. (Let for example $f_\varepsilon(x) := \exp(-1/(\varepsilon^2 - x^2))$ if $x^2 < \varepsilon^2$ and $= 0$ otherwise. Then $\delta_\varepsilon(x) := f_\varepsilon(x)/\|f_\varepsilon\|_1$, defines a positive δ_ε-sequence in $C_c^\infty(\mathbb{R})$ such that $\langle \mathbf{1}, \delta_\varepsilon\rangle = 1$ for all $\varepsilon > 0$ and $\lim_{\varepsilon\to 0} = \delta$ (= Dirac distribution). Setting $u_\varepsilon = \delta_\varepsilon^{1/2}$ we have $\mathrm{supp}(u_\varepsilon) \subset [x - \varepsilon, x + \varepsilon], x \in \mathbb{R}, \varepsilon$ arbitrary > 0 and $\|u_\varepsilon\|^2 = \langle \mathbf{1}, \delta_\varepsilon\rangle = 1$ for all $\varepsilon > 0$.)

The ‘dissipative’ action can be seen as follows. Let $\hat{\lambda}(z) \equiv \hat{\lambda}_a(z) = \exp(az^2)$ and write $\Lambda \equiv \Lambda_a$. Then

$$\lim_{a\to 0}(\Lambda_a \alpha_t \delta)(x) = \exp(-x^2/4t)/(2\sqrt{\pi t}).$$

Now, $\lim_{t->0}[\exp(-x^2/4t)/(2\sqrt{\pi t})] = \delta(x)$, hence $\lim_{a->0} \Lambda_a \delta = \delta$.

5. Intrinsic stochasticity for nonrelativistic free Hamiltonians

Let now $\mathcal{H} = L^2(\mathbb{R}^{3N})$ and $\mathcal{A} = \mathcal{L}(\mathcal{H})$. For the following calculations $\mathbb{H} = L^2(\mathcal{A}, \tau)$ is conveniently represented in the following way (cf [5]). Let $\mathbb{H}$ consist of all functions in $L^2(\mathbb{R}^{6N})$ and define

$$(fh)(x, y) = \int_{\mathbb{R}^{3N}} f(x, \xi)h(\xi, y)\,\mathrm{d}\xi, \tag{18}$$

$$\langle f, h\rangle = \int_{\mathbb{R}^{6N}} \overline{f(x, y)}h(x, y)\,\mathrm{d}(x, y), \tag{19}$$

$$f^*(x, y) = \overline{f(y, x)}. \tag{20}$$

The Fourier transform in $\mathbb{H}$ is defined by

$$(\mathcal{F}f)(k, l) \equiv \hat{f}(k, l) = (2\pi)^{-3N} \int_{\mathbb{R}^{3N}} \overline{f(x, y)}\exp[\mathrm{i}(kx - ly)]\,\mathrm{d}(x, y), \tag{21}$$

where $kx \equiv \sum_{1\leqslant j\leqslant N} \left(k_{j_1}x_{j_1} + k_{j_2}x_{j_2} + k_{j_3}x_{j_3}\right)$. The minus sign for y in $\exp[\mathrm{i}(kx - ly)]$ has been chosen because $\mathbb{H}$ is actually isomorphic to $\mathcal{H} \otimes \mathcal{H}^*$ (cf [5]). Let further

$$(Hf)(x, y) = -\sum_{1\leqslant j\leqslant N} \left(\partial^2/\partial x_{j_1}^2 + \partial^2/\partial x_{j_2}^2 + \partial^2/\partial x_{j_3}^2\right) f(x, y), \tag{22}$$

$$(H^{\times} f)(x, y) = -\sum_{1\leqslant j\leqslant N} \left(\partial^2/\partial y_{j1}^2 + \partial^2/\partial y_{j2}^2 + \partial^2/\partial y_{j3}^2\right) f(x, y), \tag{23}$$

be the left- and right-hand acting nonrelativistic free Hamiltonians, respectively. By definition H has an intrinsic stochasticity if $\tilde{\alpha}_t\Lambda = \Lambda\alpha_t$ where

$$\alpha_t = \exp[\mathrm{i}t(H - H^{\times})], \qquad \tilde{\alpha}_t = \exp[-t(H + H^{\times})], \qquad t \geqslant 0. \tag{24}$$

Theorem 3. *The nonrelativistic free Hamiltonian has an intrinsic stochasticity.*

Proof. Let $\tilde{\omega}(k, l) = k^2 + l^2$ and $\omega(k, l) = k^2 - l^2$. Then the function ϑ of lemma 1 must satisfy

$$\tilde{\omega}(k, l) = -\mathrm{i}\omega(\vartheta(k, l)). \tag{25}$$

We shall use the following solution:

$$\vartheta(k, l) = (\epsilon k, \bar{\epsilon} l), \qquad \epsilon = \mathrm{i}^{1/2}. \tag{26}$$

Defining now $\hat{\Lambda}$ according to lemma 1 we choose

$$\hat{\lambda}(z, \zeta) = \hat{c}(z, \zeta)\exp[\mathrm{i}a\omega(z, \zeta)], \qquad a = \mathrm{const} > 0, \tag{27}$$

where $\hat{c}$ is in $\mathcal{L}_\vartheta$ and grows at most exponentially. As in the proof of theorem 2 it is then shown that $C_c^\infty(\mathbb{R}^{3N})$ is in the domain of Λ and that Λ has a densely defined inverse. Since finally

$$(\hat{\Lambda}\beta_t \hat{f})(k, l) = \hat{\lambda}(\epsilon k, \bar{\epsilon} l)\hat{f}(\epsilon k, \bar{\epsilon} l)\exp\{\mathrm{i}t[(\epsilon k)^2 - (\bar{\epsilon} l)^2]\}$$

it follows that $\tilde{\alpha}_t\Lambda = \Lambda\alpha_t$ holds on $C_c^\infty(\mathbb{R}^{3N})$. This concludes the proof. □

Corollary 8. $\Lambda = \Lambda^{\times}$ *if* $\hat{c}(\vartheta(k, l)) = \overline{\hat{c}(\vartheta(l, k))}$.

Proof. Let $\hat{g}(k, l) = \hat{g}(l, k) > 0$ be a real, rapidly decreasing function. Then we can assume Λ to have an integral kernel

$$\begin{aligned}\lambda(x, y; \xi, \eta) = (2\pi)^{-6N} &\int_{\mathbb{R}^{6N}} \hat{c}(\epsilon k, \bar{\epsilon} l) \\ &\hat{g}(k, l)\exp\{-\mathrm{i}[k(x - \epsilon\xi) - l(y - \bar{\epsilon}\eta)]\}\,\mathrm{d}(k, l).\end{aligned} \tag{28}$$

By definition the *-invariance of Λ requires

$$\lambda^{\times}(x, y; \xi, \eta) \equiv \bar{\lambda}(y, x; \xi, \eta) = \lambda(x, y; \xi, \eta). \tag{29}$$

Comparing this with (26) the assertion follows. □

To demonstrate the qualitative behaviour of the temporal decay we make the following choices: $\hat{c}(\epsilon k, \bar{\epsilon} l) = 1$, $f(x, y) = \exp(-x^2 - y^2)$, $x, y \in \mathbb{R}^{3N}$ and $\hat{g}(k, l) = \exp[-k^{2n} - l^{2n}]$, $a = \mathrm{const} > 0$, where n is a positive integer $\geqslant 1$. Replacing H by $H/2m_0$, equivalently t by $t/2m_0$, $m_0 =$ rest-mass, we obtain for $n = 1$,

$$\|\tilde{\alpha}_t \Lambda f\| = c_0(1 + t/2m_0)^{-3N/2}, \qquad c_0 = \mathrm{const} > 0. \tag{30}$$

which obviously is not defined for $t \leqslant -2m_0$. For $n \geqslant 2$ we obtain for $\|\tilde{\alpha}_t\Lambda f\|$ expressions in terms of Bessel or hypergeometric functions which are strictly decreasing and defined for all $|t| < \infty$ and with an asymptotic behaviour $\sim 1/t^\beta$, $0 < \beta \leqslant 1$, $t \to \infty$. In the next section it will be shown that for the relativistic free Hamiltonian the temporal decay is completely different, namely exponentially.

Using regularization techniques it can be shown that M has an integral kernel

$$\mu(x, y; \xi, \eta) = (2\pi)^{-6N} \int_{\mathbb{R}^{6N}} |\hat{c}(\epsilon k, \bar{\epsilon} l)|^2 \hat{g}(k, l)^2 \exp\{-\mathrm{i}[k(\bar{\epsilon} x - \epsilon \xi) - l(\epsilon y - \bar{\epsilon}\eta)]\} \,\mathrm{d}(k, l). \tag{31}$$

It is evident from this expression that M is not positivity preserving in the sense that $f > 0$ as a function of $L^2(\mathbb{R}^{6N})$ implies $Mf > 0$. However, if f is viewed as an operator, that is, as an element of $\mathcal{A}$, then positivity has a different meaning. In this case $f > 0$ means that

$$(\varphi, f\varphi) := \int_{\mathbb{R}^{6N}} \overline{\varphi(x) f(x, y)} \varphi(y) \,\mathrm{d}(x, y) > 0 \tag{32}$$

for all $\varphi \in L^2(\mathbb{R}^{3N})$. It was shown in [5] that if Λ is implemented by a finite set of operators $A_1, B_1, \ldots, A_m, B_m \in \bar{\mathcal{A}}$ with domain and range in $\mathcal{H}$, meaning that $\Lambda f = A_1 f B_1 + \cdots + A_m f B_m$, then the *entropy production operator*

$$\Gamma := \mathrm{s} - \lim_{t\downarrow 0}[t^{-1}(M - \alpha_t^* M \alpha_t)] \tag{33}$$

cannot be positivity preserving.

Corollary 9. *With the hypothesis of corollary 8 M preserves positivity.*

Proof. $f \to \hat{f}$ is a unitary transformation in $\mathcal{H}$. Hence $f > 0$ implies $\hat{f} > 0$ and vice versa. Further $(\varphi, (Mf)\varphi) = (\hat{\varphi}, (\hat{M}\hat{f})\hat{\varphi})$. Now, as is easily verified,

$$(\hat{M}\hat{f})(k, l) = |\hat{\lambda}(k, l)|^2 \hat{f}(k, l). \tag{34}$$

Thus if f and hence also $\hat{f}$ is nonnegative the same is true for $\hat{M}\hat{f}$. This proves the assertion. □

Remark 3. We will briefly show that the assumption of positivity preserving of Γ would contradict that Λ is finitely implemented by $\bar{\mathcal{A}}$. Skipping a lengthy calculation one obtains

$$(\hat{\Gamma}\hat{f})(p, q) = 2\mathrm{i}(p^2 - q^2)|\hat{\lambda}(p, q)|^2 \hat{f}(p, q). \tag{35}$$

Assume that Γ and hence $\hat{\Gamma}$ preserve positivity, and that Λ is finitely implemented, that is,

$$\hat{\lambda}(p, q) = \hat{a}_1(p)\hat{b}_1(q) + \cdots + \hat{a}_m(p)\hat{b}_m(q). \tag{36}$$

Then (cf [5]) there would be a dense linear subset $D \subset L^1(\mathcal{A}, \tau)$ which is mapped by Γ into $L^1(\mathcal{A}, \tau)$ and contains strictly positive elements (this is a consequence of $L^1 L^1 = L^2$). Hence there are $h > 0$ in D for which $0 < \Gamma h \in D$. Consequently, $\langle \mathbf{1}, \Gamma h\rangle = \langle \Gamma \mathbf{1}, h\rangle > 0$, that is, $\Gamma 1 \neq 0$. But this requires by (33) that $\lim_{p\to q}(p^2 - q^2)|\hat{\lambda}(p, q)|^2 \neq 0$ which is impossible if (34) holds. (By the foregoing relation $|\hat{\lambda}(p, q)|^2$ must be a δ-type distribution with a support that is concentrated on $p = q$. An illustrating simple example is provided by an orthonormal set of real functions $f_1(p), f_2(p), \ldots$. Taking the (weak) limit $\lim_{n\to\infty} \sum_{1\leqslant j\leqslant n} f_j(p) f_j(q)$ yields $\delta(p - q)$.) Note that with the operations (18) and (19) $L^1(\mathbb{R}^{6N})$ is a representation of $L^1(\mathcal{A}, \tau)$, so that we can shortly write L^1 or L^2 without causing confusion (recall that $L^2 L^2 = L^1 \subset L^2$).

Corollary 10. *Assuming* $\hat{c}(0, 0) = 1$ *there exists a dense linear subset* $D \subset \mathrm{dom}(\Lambda) \cap L^1$ *so that* $0 < f \in D$ *implies* $\langle \mathbf{1}, \Lambda f\rangle = \langle \mathbf{1}, f\rangle > 0$.

Proof. Let (in the representation we have chosen for $\mathbb{H}$) $D = C_c^\infty(\mathbb{R}^{6N})$. Then D is a dense linear subset of L^1 which is in the domain of Λ, and a short calculation yields

$$\hat{c}(0, 0)\langle \mathbf{1}, \Lambda f\rangle = \langle \mathbf{1}, \Lambda f\rangle = \langle \mathbf{1}, f\rangle > 0.$$

□

Corollary 11. *Let* $u \in \mathcal{E}'(\mathbb{R}^{6N})$. *Then*

(a) $\|\tilde{\alpha}_t \Lambda u\| = \|\Lambda \alpha_t u\| < \infty, t \geqslant 0$;
(b) $0 < -\mathrm{d}\|\Lambda \alpha_t u\|/\mathrm{d}t < \infty, t \geqslant 0$.

Proof. (a) is equivalent to $\|\tilde{\beta}_t \hat{\Lambda} \hat{u}\| = \|\Lambda \beta_t \hat{u}\| < \infty, t \geqslant 0$, where

$$(\beta_t \hat{u})(k, l) = \exp[\mathrm{i}t(k^2 - l^2)]\hat{u}(k, l), \qquad (\tilde{\beta}_t \hat{u})(k, l) = \exp[-t(k^2 + l^2)]\hat{u}(k, l).$$

Since (cf [8]) $\hat{u}$ is entire analytic and grows at most exponentially, it follows that both β_t and $\tilde{\beta}_t$ are in the domain of Λ. This proves (a); (b) follows from (a) by the strong continuity of α and $\tilde{\alpha}$ together with the strict contractiveness of $\tilde{\alpha}$. □

Remark 4. Note that the α_t do not map $\mathcal{E}'(\mathbb{R}^{6N})$ into itself. Corollary 11 implies that there are elements $u \in \mathrm{dom}(\Lambda)$ which have sharply concentrated densities, for example, $|u(x, y)|^2 = \delta(x)\delta(y)$. The same is true for momentum densities; that is, there are $\hat{u} \in \mathrm{dom}(\hat{\Lambda})$ so that $|\hat{u}(k, l)|^2 = \delta(k)\delta(l)$. We omit a proof.

Let us finally apply statement (S3 i, appendix A) to the general case. The generator of α in the here chosen representation is $\mathrm{i}\delta_H = -\mathrm{i}(\partial^2/\partial x^2 - \partial^2/\partial y^2)$. Assume that $f_0 \in L^2(\mathbb{R}^2)$ is an eigenfunction of Γ belonging to an eigenvalue 0. Then by (S3 i) there holds $\alpha_t(M f_0) = M f_0$ from which it follows that $\delta_H(f_0)(x, y) = 0$. The general solution of this equation is $f_0(x, y) = c_1 h_1(x + y) + c_2 h_2(x - y)$ with arbitrary constants c_1 and c_2 and arbitrary twice differentiable functions (distributions) h_1 and h_2. But f_0 is clearly not in $L^2(\mathbb{R}^2)$. That also $h = M^{-1} f_0$ is stable under α, that is, satisfies $\delta_H(h) = 0$, can be seen as follows. Let $h_j(x \pm y) = \int_{\mathbb{R}} \hat{h}(z) \exp[-\mathrm{i}(x - y)z]\, \mathrm{d}z$ and write $w_z^{\pm}(x, y) \equiv \exp[-\mathrm{i}(x \pm y)z]$. Using the integral representation for M^{-1} we have $\left(M^{-1} w_z^{\pm}\right)(x, y) = 2 \exp[-z(x \mp y)]/\pi^2$. Hence

$$\delta_H\left(M^{-1} w_z^{\pm}(x, y)\right) = (2/\pi^2) \int_{\mathbb{R}} \hat{h}(z) \delta_H(\exp[-z(x \mp y)])\, \mathrm{d}z = 0$$

and therefore $\alpha_t(M^{-1} f_0) = M^{-1} f_0, t \geqslant 0$, and thus $\Gamma(M^{-1} f_0) = 0$. That is, $M^{-1} f_0 \notin L^2(\mathbb{R}^2)$ is a generalized eigenfunction of Γ belonging to an eigenvalue 0. So by (S4, appendix A) M has an empty point spectrum (that δ_H is absolutely continuous by (S3, appendix A) is already obvious by its representation as a hyperbolic differential operator).

6. Intrinsic stochasticity for relativistic free Hamiltonians

Let $\mathcal{A} = \mathcal{L}(\mathbb{F}_s \otimes \mathbb{F}_s^*)$, where $\mathbb{F}_s$ is a symmetric Fock space and the direct sum of symmetric n-particle spaces $\mathbb{F}_s^{(n)} = L_s^2(\mathbb{R}^{3n})$. For $\mathbb{H} = L^2(\mathcal{A}, \mathrm{trace})$ we use the representation of the preceding section with $\mathcal{H} = L^2(\mathbb{R}^{3n})$ replaced by $\mathbb{F}_s$. That is, $\mathbb{H} = \oplus_{m,n=1}^{\infty} \mathbb{F}^{(m,n)}$ where $\mathbb{F}^{(m,n)} = \mathbb{F}_s^{(m)} \otimes \mathbb{F}_s^{(n)*}$. The momentum representation $\hat{H}$ of a relativistic free Hamiltonian with mass m_0 is given by

$$(\hat{H}\hat{\phi})^{(n)}(k) = \hat{h}^{(n)}(k)\hat{\phi}^{(n)}(k) = \sum_{p=1}^{n} \mu(k_p)\hat{\phi}^{(n)}(k), \tag{37}$$

where $\mu(k_p) = \left(m_0^2 + k_p^2\right)^{1/2}, k_p \in \mathbb{R}^3$. Thus for $\hat{\phi}^{(m,n)} \in \mathbb{F}^{(m,n)}$

$$(\beta_t \hat{\phi})^{(m,n)}(k, l) = \exp\{\mathrm{i}t[\hat{h}^{(m)}(k) - \hat{h}^{(n)}(l)]\}\hat{\phi}^{(m,n)}(k, l), \tag{38}$$

$$(\tilde{\beta}_t \hat{\phi})^{(m,n)}(k, l) = \exp\{-t[\hat{h}^{(m)}(k) + \hat{h}^{(n)}(l)]\}\hat{\phi}^{(m,n)}(k, l). \tag{39}$$

Let us assume that $\hat{\Lambda}$ is reduced by each $\mathbb{F}^{(m,n)}$ so that $\hat{\phi} \in \mathrm{dom}(\hat{\Lambda}) \cap \mathbb{F}^{(m,n)}$ implies $\hat{\Lambda} \in \mathbb{F}^{(m,n)}$. That is, relation (1) is assumed to hold on a dense linear subset $D^{(m,n)}$ of $\mathbb{F}^{(m,n)}$ for arbitrary natural m, n. It follows then from (36) and (37) that the function ϑ of lemma 1 can be chosen

$$\begin{aligned} &\vartheta(k,l) = (\nu(k), \overline{\nu(l)}) = (\nu(k)_1, \ldots, \nu(k)_m, \overline{\nu(l)}_1, \ldots, \overline{\nu(l)}_n), \\ &\nu(k)_p = \mathrm{i}\big(2m_0^2 + k_p^2\big)^{1/2}, \qquad k_p \in \mathbb{R}^3 \end{aligned} \tag{40}$$

where $\mu(\nu(k)_p)$, $1 \leqslant p \leqslant m$, and $\mu(\nu(l)_q)$, $1 \leqslant q \leqslant n$, belong to different branches of the root. That is, $\mu(\nu(k)_p) = \mathrm{i}\mu(k_p)$, and $\mu(\nu(l)_q) = -\mathrm{i}\mu(l_q)$. Let

$$\hat{\lambda}(z,\zeta) = \hat{c}(z,\zeta)\exp[a(z^2+\zeta^2)], \qquad a = \text{const} > 0, \tag{41}$$

where $\hat{c}$ is in $\mathcal{L}_\vartheta$ and grows at most exponentially, and $\hat{c}(\vartheta(k,l))$ is separately symmetric in k and l. Let $\phi \in D^{(m,n)} = \mathbb{F}^{(m,n)} \cap C_c^\infty(\mathbb{R}^{3m} \times \mathbb{R}^{3n})$. Then (cf [8, 10]) its Fourier–Laplace transform is entire analytic and grows at most exponentially. Thus $(\hat{\Lambda}\hat{\phi})(k,l) = \hat{\lambda}(\vartheta(k,l))\hat{\phi}(\vartheta(k,l))$ implies by (40) and (41) that $\hat{\phi} \in \mathrm{dom}(\hat{\Lambda})$ and hence $\phi \in \mathrm{dom}(\Lambda)$. Since $D^{(m,n)}$ is dense in $\mathbb{F}^{(m,n)}$ it follows that (1) holds on a dense subset of $\mathbb{F}^{(m,n)}$. The inverse of $\hat{\Lambda}$ (on $\mathbb{F}^{(m,n)}$) is constructed in the fashion of lemma 2. By (39) it suffices thereby to assume that the function $(k,l) \to \hat{c}(k,l)^{-1}$ is analytic in a neighbourhood of each $(k,l) \in \mathbb{R}^{3m} \times \mathbb{R}^{3n}$ and grows at most exponentially. We have just proved

Theorem 4. *Let $H = \big(-\Delta + m_0^2\big)^{1/2}$ be the relativistic free Hamiltonian of a particle with mass m_0. Then H has an intrinsic stochasticity if Λ is reduced by each subspace $\mathbb{F}^{(m,n)}$ of $\mathbb{H} = \oplus_{m,n=1}^\infty \mathbb{F}^{(m,n)}$.*

A short calculation yields for Λ restricted to $\mathbb{F}^{(m,n)}$ the following integral kernel:

$$\begin{aligned} \lambda(x,y;\xi,\eta) = (2\pi)^{-3(m+n)/2} \exp\big[-2a(m+n)m_0^2\big] \int_{\mathbb{R}^{3m}\times\mathbb{R}^{3n}} \hat{c}(\vartheta(k,l)) \\ \times \exp\{-a(k^2+l^2) - i[kx - ly - \nu(k)\xi + \nu(l)\eta]\}\,\mathrm{d}(k,l). \end{aligned} \tag{42}$$

It follows from this relation and (28)

Corollary 12. $\Lambda = \Lambda^\times$ *if* $\hat{c}(\vartheta(k,l)) = \overline{\hat{c}(\vartheta(k,l))}$.

Applying regularization techniques one obtains from (41) the following integral kernel for $M = \Lambda^*\Lambda$:

$$\begin{aligned} \rho(x,y;\xi,\eta) = (2\pi)^{-3(m+n)/2} \exp\big[-4a(m+n)m_0^2\big] \int_{\mathbb{R}^{3m}\times\mathbb{R}^{3n}} |\hat{c}(\vartheta(k,l))|^2 \\ \times \exp[-2a(k^2+l^2) + |\nu|(k)(x-\xi) + |\nu|(l)(y-\eta)]\,\mathrm{d}(k,l), \end{aligned} \tag{43}$$

where $|\nu|(k) \equiv (|\nu(k)_1|, \ldots, |\nu(k)_m|)$. Clearly, $\rho > 0$, hence there holds

Corollary 13. *M is positivity preserving w.r.t. real-valued positive functions as well as to positive operators in* $\mathrm{dom}(M) \cap \mathcal{A}$.

Corollary 14. *Let $u \in \mathcal{E}'(\mathbb{R}^{3m} \times \mathbb{R}^{3n})$. Then*

(*a*) $\|\tilde{\alpha}_t \Lambda u\| = \|\Lambda \alpha_t u\| < \infty, \qquad t \geqslant 0;$

(*b*) $0 < -d\|\Lambda\alpha_t u\|/\mathrm{d}t < \infty, \qquad t \geqslant 0.$

The proof is completely analogous to that of corollary 11 in the previous section.

By a suitable normalization factor for the function $\hat{c}$ we can again achieve that $\langle \mathbf{1}, \Lambda f\rangle = \langle \mathbf{1}, f\rangle$ for all $f \in \mathrm{dom}(\Lambda)$.

Remark 5. There is a striking difference between the relativistic and nonrelativistic case with regard to the mass m_0. In the nonrelativistic case we have actually the Hamiltonian H to divide by $2m_0$ (setting $\hbar = 1$). This has basically no consequences for Λ, as can be easily verified. However, since now $\tilde{\alpha}_t = \exp[-t(H + H^\times)/2m_0]$ we have $\partial\|\tilde{\alpha}_t\Lambda f\|/\partial m_0 > 0$ and $\|\tilde{\alpha}_t\Lambda f\| \to \|\Lambda f\|$ for $m_0 \to \infty$. In the relativistic case Λ depends on m_0 as can be seen from (41). Using a momentum representation it is no great difficulty to show that $\partial\|\tilde{\alpha}_t\Lambda f\|/\partial m_0 < 0$ and $\|\tilde{\alpha}_t\Lambda f\| \to 0$ for $m_0 \to \infty$. That is, the decay rate increases here with the mass m_0 contrary to the behaviour in the nonrelativistic case. A more detailed information will be provided by the following example.

Example. Let $\hat{c}(\vartheta(k,l)) = 1$. In order to calculate $\|\tilde{\alpha}_t\Lambda f\|^2$ for an $f \in \mathbb{F}^{(m,n)}$ it suffices in a first step to let $f \in \mathbb{F}_s^{(1)}$ because if $f_{(m)}$ is the m-fold tensorproduct of f and $\bar{f}_{(n)}$ the n-fold tensorproduct of $\bar{f} \in \mathbb{F}_s^{(1)*}$, and if $f_{(m,n)} = f_{(m)} \otimes \bar{f}_{(n)}$ then $\|\tilde{\alpha}_t\Lambda f_{(m,n)}\|^2 = \|\tilde{\alpha}_t\Lambda f\|^{2(m+n)}$ as can be easily shown (by setting $\hat{c}(\vartheta(k,l)) = 1$ we have no intertwining between the state spaces whose tensorproduct yields $\mathbb{F}^{(m,n)}$). For an example that leads to an explicit result we let f be the Fourier transform of the function $g(p;b) = \exp(-bp^2)\left(2m_0^2 + p^2\right)^{-1/4}$, where b is a positive constant and $p^2 \equiv |p|^2 = p_1^2 + p_2^2 + p_3^2$, $p = \{p_1, p_2, p_3\} \in \mathbb{R}^3$ (assuming thus radial symmetry). Clearly this function and hence its Fourier transform is in $L^2(\mathbb{R}^3)$. Skipping a lengthy calculation one obtains

$$\|\tilde{\alpha}_t\Lambda f\|^2 = (4\pi/8A^{3/2})\exp\left[-2m_0\left(Am_0^2 + t\right) - 2Am0^2\right] \times \left\{2\sqrt{A} + \sqrt{2\pi}\,t\exp[(2Am_0 + t)^2]\left[-1 + \Phi\left(\frac{2Am_0 + t}{\sqrt{2A}}\right)\right]\right\}, \tag{44}$$

where $A = a - b > 0$ and Φ denotes the error function. Note that b can be used to generate a set of linearly independent functions $p^{2n}g(p,b) = (-1)^n\partial_b^n g(p,b)$. The asymptotic behaviour of (43) is

$$\|\tilde{\alpha}_t\Lambda f\|^2 = \pi\exp[-2m_0(2Am_0 + t)]\left[2m_0/t + \left(1 - 4Am_0^2\right)/t^2 + \left(4Am_0 + 168A^2m_0^3 - 10Am_0\left(1 + 16Am_0^2\right)\right)/t^3 + O(1/t^4)\right]. \tag{45}$$

Remark 6. The preceding examples of free nonrelativistic and relativistic Hamiltonians have shown that an exponentially contractive semi-group of maps $\tilde{\alpha}_t = \exp[-t(H + H^\times)]$ does not necessarily imply an exponential temporal behaviour of states. Whereas in the nonrelativistic case we had $\|\tilde{\alpha}_t\Lambda f\|^2 \sim (2m_0/t)^{3/2}$ the relativistic Hamiltonian had delivered $\|\tilde{\alpha}_t\Lambda f\|^2 \sim \exp(-2m_0 t)$, $m_0 =$ restmass,$1 \ll m_0 t$.

7. Intrinsic stochasticity of a boson spin-zero particle–antiparticle system

It was shown in [5] that the operation $T \to T^\times$ can be related to a particle–antiparticle scheme. We shall reconsider this interpretation in the following. Let $\mathcal{A} = \mathcal{L}(\mathcal{H})$, $\mathcal{H} = L^2(\mathbb{R})$, and let (p,q) be a Heisenberg couple acting in $\mathbb{H} = L^2(\mathcal{A}, \text{trace})$. Let further $\Theta = (\Theta_{jk})$ be a real constant 2×2-matrix for which $\Theta_{12}\Theta_{21} + \Theta_{11}\Theta_{22} = 1$, and let $P = \Theta_{11}p + \Theta_{12}p^\times$, $Q = \Theta_{22}q + \Theta_{21}q^\times$, where $p^\times f := fp$, $q^\times f := fq$ (according to the definition given above). Then $\mathrm{i}[P,Q] \equiv \mathrm{i}(PQ - QP) = \mathbf{1}$ (setting $\hbar = 1$ and taking into account that $[p^\times, q] = [q^\times, p] = 0$). Hence (P,Q), too, is a Heisenberg couple in $\mathbb{H} = L^2(\mathcal{A}, \text{trace})$. Since $(\gamma ST)^\times = \bar{\gamma}S^\times T^\times$, $\gamma \in \mathbb{C}$, for any two linear maps in $\mathbb{H}$ there holds $\mathbf{1}^\times = \mathbf{1} = (\mathrm{i}[P,Q])^\times = -i[P^\times, Q^\times]$. Thus either P or Q but not both can be *-invariant. Since the momentum of a freely moving particle (as a function of time) has odd time parity, we let $P^\times = -P$. We are now going to consider a transformation $T \to T^\times$ more general as part of a compound system consisting of particles and antiparticles. A simple example is that of a harmonic oscillator with a Hamiltonian $\mathbf{H} = (P^2 + Q^2)/2$. Let $a = (q - \mathrm{i}p)/\sqrt{2}$ and $a^* = (q + \mathrm{i}p)/\sqrt{2}$ be the annihilation and creation operators respectively for particles and $b \equiv a^\times = (q^\times - \mathrm{i}p^\times)/\sqrt{2}$ and $b^* \equiv a^{*\times} = (q^\times + \mathrm{i}p^\times)/\sqrt{2}$ the corresponding operators for antiparticles. Then $H = a^*a + 1/2$ and $H^\times = b^*b + 1/2$ are the Hamiltonians of a harmonic particle and antiparticle oscillator, respectively. Let $\Theta_{11} = \Theta_{22} = \cos\theta$, $\Theta_{12} = \Theta_{21} = \sin\theta$. Then

$$\mathbf{H} = \cos^2\theta H + \sin^2\theta H^\times + \sin(2\theta)W/2, \tag{46}$$

where

$$W = pp^\times + qq^\times = a^*b + b^*a$$

describes a particle–antiparticle interaction with a *mixing parameter* θ.

The annihilation and creation operators of $\mathbf{H}$ are $A = (Q - \mathrm{i}P)/\sqrt{2}$ and $A^* = (Q + \mathrm{i}P)/\sqrt{2}$. The orthonormal eigenstates of the number operator $\mathbf{N} = A^*A$ and $\mathbf{H} = A^*A + 1/2$ are in a representation $\mathbb{H} = \mathcal{H} \otimes \mathcal{H}^*$

$$\Psi_n = \sum_{j=0}^{n} \binom{n}{j}^{1/2} \cos^{n-j}\theta \sin^j\theta \psi_{n-j} \otimes \bar{\psi}_j^\times, \tag{47}$$

that is, $A^*\Psi_n = (n+1)^{1/2}\Psi_{n+1}, \mathbf{N}\Psi_n = n\Psi_n$. The ψ_j and $\bar{\psi}_j^\times$ represent the particle and antiparticle states, respectively. That is

$$a^*\psi_n = (n+1)^{1/2}\psi_{n+1}, \qquad a^*a\psi_n = n\psi_n,$$
$$b^*\bar{\psi}_n^\times = (n+1)^{1/2}\bar{\psi}_{n+1}, \qquad b^*b\bar{\psi}_n = n\bar{\psi}_n^\times.$$

If we extend this scheme to a Klein–Gordon field with mass m_0 represented by a canonical couple $\{\varphi(x), \pi(x)\}$ for particles and $\{\varphi(x)^\times, \pi(x)^\times\}$ for antiparticles and if $\{\Phi(x) = \cos\theta\varphi(x) + \sin\theta\varphi(x)^\times, \Pi(x) = \cos\theta\pi(x) + \sin\theta\pi(x)^\times\}$ represents the compound field, then the (Wick ordered) Hamiltonian of the Φ-field is

$$\begin{aligned}\mathbf{H} &= \int_{\mathbb{R}^3} : \left[\Pi^2(x) + (\nabla\Phi)^2(x) + m_0^2\Phi^2(x)\right] : \mathrm{d}x/2 \\ &= \cos^2\theta H + \sin^2\theta H^\times + \sin(2\theta)W/2,\end{aligned} \tag{48}$$

where H and $H^\times$ are the Hamiltonians of the φ- and $\varphi^\times$-field, respectively, that is,

$$H = \int_{\mathbb{R}^3} \mu(k)a^*(k)a(k)\,\mathrm{d}k, \qquad \mu(k) = \left(m_0^2 + k^2\right)^{1/2}, \tag{49}$$

$$H^\times = \int_{\mathbb{R}^3} \mu(l)b^*(l)b(l)\,\mathrm{d}l, \qquad \mu(l) = \left(m_0^2 + l^2\right)^{1/2}, \tag{50}$$

and

$$W = \int_{\mathbb{R}^3} \mu(k)[a^*(k)b(-k) + b^*(-k)a(k)]\,\mathrm{d}k. \tag{51}$$

$a(k)$ and $b(k) = a(k)^\times$ are the annihilation operators of the φ- and $\varphi^\times$-field, respectively. As to an intrinsic stochasticity we have for the generators of α and $\tilde{\alpha}$ respectively

$$\mathrm{i}\delta_{\mathbf{H}} = \mathrm{i}(\mathbf{H} - \mathbf{H}^\times) = \mathrm{i}\cos(2\theta)(H - H^\times), \tag{52}$$

$$\tilde{\delta}_{\mathbf{H}} = \mathbf{H} + \mathbf{H}^\times = H + H^\times + \sin(2\theta)W. \tag{53}$$

The annihilation operator for the Φ-field is $A_\theta(p) = \cos\theta a(p) + \sin\theta b(-p)$ and the eigenstates of the number operator $A_\theta^*(p)A_\theta(p)$ are the functions

$$\Psi_\theta^{(n)}(k_1, \ldots, k_n) = \sum_{j=0}^{n} \binom{n}{j}^{1/2} \cos^{n-j}\theta \sin^j\theta \\ \times \psi^{(n-j)}(k_1, \ldots, k_{n-j}) \otimes \bar{\psi}^{(j)\times}(k_{n-j+1}, \ldots, k_n). \tag{54}$$

That is, if $\mathbf{N}_\theta = \int_{\mathbb{R}^3} A_\theta^*(p)A_\theta(p)\,\mathrm{d}p$ then

$$\mathbf{N}_\theta \Psi_\theta^{(n)}(k_1, \ldots, k_n) = n\Psi_\theta^{(n)}(k_1, \ldots, k_n), \tag{55}$$

$$\mathbf{H}\Psi_\theta^{(n)}(k_1, \ldots, k_n) = \sum_{i=1}^{n} \mu(k_i)\Psi_\theta^{(n)}(k_1, \ldots, k_n). \tag{56}$$

Note that $B_\theta(p) = A_{\pi/2-\theta}^\times(-p)$.

If we let $\theta = \pi/4$ or $-\pi/4$ so that $\{\Pi^\times, \Phi^\times\} = \{-\Pi, \Phi\}$ or $\{\Pi, -\Phi\}$ respectively then $\delta_{\mathbf{H}} = 0$, meaning that we have no dynamics. Now, $\theta = \pm\pi/4$ implies $A(k) = a(k) \pm b(-k)$, that is, a symmetry between particles and antiparticles. In other words: dynamics and a Lyapunow process viz intrinsic stochasticity in the here considered compound-model are simultaneously possible only if there is an asymmetry between the number of particles and antiparticles. This could explain why there are more particles than antiparticles (or vice versa) observed in connection with an arrow of time.

Remark 7. Note that the *-invariant term W does not appear in the dynamical part $\mathrm{i}\delta_{\mathbf{H}}$.

It was shown that $\Psi_\theta^{(n)}$ as an element of $\mathbb{S}_{\mathbb{F}}^{(n)} \equiv \oplus_{j=0}^{n}\mathbb{F}^{(n-j,j)}$ is mapped by $A_\theta(p)$ into $\mathbb{S}_{\mathbb{F}}^{(n+1)}$. An easy calculation proves that generally

$$A_\theta(p)\mathbb{S}_{\mathbb{F}}^{(n)} \subset \mathbb{S}_{\mathbb{F}}^{(n+1)}, \quad \mathbf{N}_\theta\mathbb{S}_{\mathbb{F}}^{(n)} \subset \mathbb{S}_{\mathbb{F}}^{(n)}, \quad \mathbf{H}\mathbb{S}_{\mathbb{F}}^{(n)} \subset \mathbb{S}_{\mathbb{F}}^{(n)}, \quad \mathbf{H}^*\mathbb{S}_{\mathbb{F}}^{(n)} \subset \mathbb{S}_{\mathbb{F}}^{(n)}. \tag{57}$$

Thus $\tilde{\delta}_{\mathbf{H}}$ and $\delta_{\mathbf{H}}$ map $\mathbb{S}_{\mathbb{F}}^{(n)}$ into itself. This justifies to require that also Λ maps a subset of $\mathbb{S}_{\mathbb{F}}^{(n)}$ into this space (in [5] we had ad hoc for technical convenience assumed that Λ is reduced by each $\mathbb{F}^{(m,n)}$). We are going to show now that we may indeed assume that Λ maps a subspace of $\mathbb{F}^{(n-j,j)}$, $j \leqslant n$, $n = 0, 1, 2, \ldots$, into itself. Writing $h^{(m,n)}(k) = \sum_{i=m}^{n}\mu(k_i)$, $h^{(0,0)}(k) = 0$, it follows from $\mathbf{H} = \cos^2\theta H + \sin^2\theta H^\times + \sin(2\theta)W/2$:

$$\sin(2\theta)W\Psi_\theta(n)(k_1 \ldots, k_n)/2 = (\mathbf{H} - \cos^2\theta H - \sin^2\theta H^\times)\Psi_\theta(n)(k_1 \ldots, k_n) \\ = \sum_{j=0}^{n}\binom{n}{j}^{1/2}\cos^{n-j}\theta\sin^j\theta(h^{(1,n)}(k) - h^{(1,n-j)}(k)\cos^2\theta - h^{(n-j+1,n)}(k)\sin^2\theta) \\ \times \psi^{(n-j)}(k_1, \ldots, k_{n-j}) \otimes \bar{\psi}^{(j)\times}(k_{n-j+1}, \ldots, k_n).$$

Writing $\psi^{(n-j,j)} \equiv \psi^{(n-j)} \otimes \bar{\psi}^{(j)\times}$ and assuming the $\psi^{(n-j,j)}$ to be orthogonal it follows for $0 \neq \theta \neq \pi/2$,

$$\sin(2\theta)W\Psi_\theta(n)\psi^{(n-j,j)} = 2(h^{(1,n-j)} - h^{(1,n-j)}\cos^2\theta - h^{(n-j+1,n)}\sin^2\theta)\psi^{(n-j,j)}.$$

Writing now $\psi^{(n-j,j)}(k,l) \equiv \psi^{(n-j)}(k_1, \ldots, k_{n-j}) \otimes \bar{\psi}^{(j)\times}(k_{n-j+1}, \ldots, k_n)$, $l_r = k_{n-j+r}$, one obtains after a short calculation

$$\tilde{\delta}_{\mathbf{H}}\psi^{(n-j,j)}(k,l) = (H + H^\times + \sin(2\theta)W)\psi^{(n-j,j)}(k,l) \\ = (h^{(1,n)}(k) + 2h^{(1,n-j)}(k) - 2h^{(1,n-j)}(k)\cos^2\theta \\ - 2h^{(n-j+1,n)}(k)\sin^2\theta)\psi^{(n-j,j)}(k,l), \tag{58}$$

$$\begin{aligned}\delta_{\mathbf{H}}\psi^{(n-j,j)}(k,l) &= \cos(2\theta)(H-H^{\times})\psi^{(n-j,j)}(k,l)\\ &= \cos(2\theta)(h^{(1,n-j)}(k)-h^{(n-j+1,j)}(k))\psi^{(n-j,j)}(k,l).\end{aligned} \tag{59}$$

so that

$$\begin{aligned}\tilde{\alpha}_t\psi^{(n-j,j)}(k,l) &= \exp[-t(H+H^{\times}+\sin(2\theta)W)]\psi^{(n-j,j)}(k,l)\\ &= \exp[-t(h^{(1,n)}(k)+2h^{(1,n-j)}(k)-2h^{(1,n-j)}(k)\cos^2\theta\\ &\quad -2h^{(n-j+1,n)}(k)\sin^2\theta)]\psi^{(n-j,j)}(k,l),\end{aligned} \tag{60}$$

$$\begin{aligned}\alpha_t\psi^{(n-j,j)}(k,l) &= \exp[\mathrm{i}\cos(2\theta)t(H-H^{\times})]\psi^{(n-j,j)}(k,l)\\ &= \exp[\mathrm{i}\cos(2\theta)t(h^{(1,n-j)}(k)-h^{(1j)}(k))]\psi^{(n-j,j)}(k,l).\end{aligned} \tag{61}$$

These relations justify to assume that Λ maps a domain which is a subset of the subspace $\mathbb{F}^{(n-j,j)}$ into this subspace. Choosing again as an example $f(p,q)=f(p)\otimes\bar{f}^{\times}(q)$ where $f=\bar{f}^{\times}$ is the Fourier transform of $g(p)=\exp(-bp^2)/\left[1+\cos^2(2\theta)m_0^2+p^2\right]^{1/4}$, and copying the corresponding calculations in section 6, we obtain $\|\tilde{\alpha}_t\Lambda f\|^2=F_1(t)F_2(t)$, where

$$\begin{aligned}F_j(t) \equiv \|\tilde{\alpha}_t\Lambda f\|^2 &= 4\pi\exp\left[-8(a-b_1)m_0^2-m_0c_jt\right]\\ &\quad\times\left\{2\sqrt{a-b_1}-\sqrt{2\pi}c_jt\exp\{-[2(a-b_1)m_0+c_jt]^2\}\right.\\ &\quad\left.\times\left[1-\Phi\left(\frac{2(a-b_1)m_0+c_jt}{\sqrt{2(a-b_1)}}\right)\right]\right\}\end{aligned} \tag{62}$$

and $b_1=b\cos(2\theta)$, $c_1=2-\cos(2\theta)$, $c_2=2+\cos(2\theta)$. The constants a,b are arbitrary up to the condition $0<a<b\cos(2\theta)$. Asymptotic behaviour is given by

$$\begin{aligned}F_j(t) &\approx 16\pi(a-b_1)\exp\left[-8(a-b_1)m_0^2-m_0c_jt\right]\\ &\quad\times\left\{1-\frac{c_1t}{c_jt+2(a-b_1)m_0}\left[1-\frac{a-b_1}{(2(a-b_1)m_0+c_jt)^2}\right.\right.\\ &\quad\left.\left.+\,\mathrm{o}([2(a-b_1)m_0+c_jt)]^{-4})\right]\right\}.\end{aligned} \tag{63}$$

8. Microscopic entropy

The function

$$(t,f)\to\sigma_f(t):=-\ln(\langle\alpha_tf,M\alpha_tf\rangle/\langle f,Mf\rangle),\qquad t\geqslant 0,$$

is positive and nondecreasing. Further, let $\mathbb{H}=\mathbb{H}_1\otimes\cdots\otimes\mathbb{H}_N$, $f=f_1\otimes\cdots\otimes f_N$, $f_j\in\mathbb{H}_j$, $\alpha_t=\alpha_t^{(1)}\otimes\cdots\otimes\alpha_t^{(N)}$, $\alpha_t^{(j)}=\alpha_t\uparrow\mathbb{H}_j$, $M=M_1\otimes\cdots\otimes M_N$, $M_j=M\uparrow\mathbb{H}_j$, and $M(t)=\alpha_t^*M\alpha_t$. Then $\langle f,M(t)f\rangle=\langle f_1,M_1(t)f_1\rangle\cdots\langle f_N,M_N(t)f_N\rangle$ and hence $\sigma_f=\sigma_{f_1}+\cdots+\sigma_{f_N}$. Because of these properties we had called σ_f an *entropy function* associated with $\{\mathbb{H},\alpha,M\}$. In the context of this definition we had introduced an *entropy production operator* Γ, defined by $\alpha_t^*\Gamma\alpha_tf:=-d(\alpha_t^*M\alpha_t)f/\mathrm{d}t$, $t\geqslant 0$, where f is assumed to be in a core $D_M\subset\mathbb{H}$ which is invariant under α.

Let N and $N^{\times}$ be number operators acting in $\mathbb{F}^{(n)}$ (see section 7) and $\mathbb{F}_s^{(n)*}$ respectively. That is, $N\psi^{(n)}=n\psi^{(n)}$ and $N^{\times}\psi^{(n)\times}=n\psi^{(n)\times}$. Let $\mathcal{N}=N+N^{\times}$ so that $\mathcal{N}\psi^{(m,n)}=(m+n)\psi^{(m,n)}$. Hence for any expression $F(n)=\sum_{j=0}^{n}c_j\psi^{(n-j,j)}$ with arbitrary constants c_j we have $\mathcal{N}F(n)=nF(n)$ and in particular $\mathcal{N}\Psi^{(n)}=n\Psi^{(n)}$. Let $\mathbb{S}_{\mathbb{F}}^{(n)}=\oplus_{j=0}^{n}\mathbb{F}^{(n-j,j)}$. By relations (59) and (60) $\tilde{\alpha}_t$ and α_t map $\mathbb{S}_{\mathbb{F}}^{(n)}$ into itself, so that $\mathcal{N}\tilde{\alpha}_t=\tilde{\alpha}_t\mathcal{N}$ and $\mathcal{N}\alpha_t=\alpha_t\mathcal{N}$.

Let $\Lambda_t = \alpha_t^* \Lambda \alpha_t$. Then Λ_t maps each subset of $\mathbb{S}_{\mathbb{F}}^{(n)}$ which is in its domain into $\mathbb{S}_{\mathbb{F}}^{(n)}$. If Φ is an element of $\mathbb{S}_{\mathbb{F}}^{(n)}$ which is in the domain of Λ_t then $\mathcal{N}\Lambda_t\Phi = n\Lambda_t\Phi$, that is, $[\mathcal{N}, \Lambda_t] = 0$ (on the domain of Λ_t). Let $M_t = \Lambda_t^*\Lambda_t = \alpha_t^* M \alpha_t$ and

$$\begin{aligned} \mathcal{N}_t^- &= \mathcal{N} + c\Lambda_t^*\mathcal{N}\Lambda_t = N(1 + cM_t), \\ \mathcal{N}_t^+ &= \mathcal{N} - c\Lambda_t^*\mathcal{N}\Lambda_t = N(1 - cM_t) \end{aligned} \tag{64}$$

where c is a positive constant. Then

$$\mathcal{N} = \left(\mathcal{N}_t^- + \mathcal{N}_t^+\right)/2.$$

Let further

$$\rho_t^{\pm} = \phi^*\mathcal{N}\phi \mp c_\phi \phi^* \Lambda_t^* \mathcal{N}\Lambda_t\phi, \tag{65}$$

where ϕ is in the domain of $\mathcal{N} \cap \Lambda_t^*\mathcal{N}\Lambda_t$ for all $t \geqslant 0$ and c_ϕ is a constant that guarantees $\rho_0^+ > 0$ (let $c_\phi < (\langle\phi, N\phi\rangle/\langle\Lambda\phi, N\Lambda\phi\rangle)$). Since $\mathcal{N}$ is a nonnegative operator which commutes with Λ_t we have

$$\begin{aligned} \tau(\phi^*\Lambda_t^*\mathcal{N}\Lambda_t\phi) &= \langle \Lambda\alpha_t\phi, \mathcal{N}\Lambda\alpha_t\phi\rangle \\ &= \langle \Lambda\alpha_t\mathcal{N}^{1/2}\phi, \Lambda\alpha_t\mathcal{N}^{1/2}\phi\rangle \\ &= \langle \tilde{\alpha}_t\Lambda\mathcal{N}^{1/2}\phi, \tilde{\alpha}_t\Lambda\mathcal{N}^{1/2}\phi\rangle. \end{aligned} \tag{66}$$

Hence the function $t \to \tau(\phi^*\Lambda_t^*\mathcal{N}\Lambda_t\phi)$ is monotonously decreasing. Thus $0 < \rho_t^+ \leqslant \rho_0^+$, $t \leqslant 0$, and $\rho \equiv \phi^*\mathcal{N}\phi = \lim_{t\to\infty}\rho_t^- = \lim_{t\to\infty}\rho_t^+$. We define now an increasing resp. decreasing intrinsic microscopic nonequilibrium entropy by the standard entropy functions

$$S_q\left(\rho_t^{\pm}\right) = \log\left\{\tau\left[\left(\rho_t^{\pm}\right)^q\right]\right\}, \qquad 0 < q \in \mathbb{R}, \tag{67}$$

$$= -\tau\left[\rho_t^{\pm}\log\left(\rho_t^{\pm}\right)\right], \qquad q = 0. \tag{68}$$

The equilibrium entropy is then

$$S_q(\rho) = \lim_{t\to\infty} S_q\left(\rho_t^-\right) = \lim_{t\to\infty} S_q\left(\rho_t^+\right). \tag{69}$$

Remark 8. Note that if we set $t = 1/\text{Temp}$ where Temp means temperature then $\exp(-tH) = \exp(-H/\text{Temp})$ viz $\exp[-t(H + H^\times)] = \exp[-(H + H^\times)/\text{Temp}]$ can be interpreted as a *Gibbs factor*.

9. Intrinsic stochasticity of the dirac equation

When trying to define a canonical intrinsic stochasticity for the Dirac equation one encounters a well-known obstacle, namely that its Hamiltonian,

$$H = -\mathrm{i}\vec{\alpha}\nabla + m_0\beta,$$

is indefinite. Let $H_+ \geqslant 0$ and $H_- \leqslant 0$ be the positive and negative part of H respectively. To calculate H_+ and H_- we use again the Fourier transformation scheme of section 1.2 (that is, a momentum representation) by replacing the partial derivatives $\partial/\partial x_j$ by multiplication operators $\mathrm{i}k_j$, $0 \leqslant j \leqslant 3$. This means that H is replaced by the matrix

$$\hat{H} = \begin{pmatrix} m_0 & 0 & k_3 & k_1 - \mathrm{i}k_3 \\ 0 & m_0 & k_1 + \mathrm{i}k_2 & -k_3 \\ k_3 & k_1 - \mathrm{i}k_3 & -m_0 & 0 \\ k_1 + \mathrm{i}k_2 & -k_3 & 0 & -m_0 \end{pmatrix}. \tag{70}$$

The eigenvalues of $\hat{H}$ are $-\sqrt{k^2+m_0^2}$ and $\sqrt{k^2+m_0^2}$, $k^2 = k_1^2+k_2^2+k_3^2$, each with multiplicity 2. Using the Dunford–Schwartz calculus (cf [12]) we can calculate the projection operators $\hat{P}_-$ and $\hat{P}_+$:

$$\hat{P}_\pm = \begin{pmatrix} \frac{\mp m_0+1}{2\sqrt{k^2+m_0^2}} & 0 & \frac{\mp k_3}{2\sqrt{k^2+m_0^2}} & \frac{\mp(k_1-\mathrm{i}k_2)}{2\sqrt{k^2+m_0^2}} \\ 0 & \frac{\mp m_0+1}{2\sqrt{k^2+m_0^2}} & \frac{\mp(k_1+\mathrm{i}k_2)}{2\sqrt{k^2+m_0^2}} & \frac{\pm k_3}{2\sqrt{k^2+m_0^2}} \\ \frac{\mp k_3}{2\sqrt{k^2+m_0^2}} & \frac{\mp(k_1-\mathrm{i}k_2)}{2\sqrt{k^2+m_0^2}} & \frac{\pm m_0+1}{2\sqrt{k^2+m_0^2}} & 0 \\ \frac{\mp(k_1+\mathrm{i}k_2)}{2\sqrt{k^2+m_0^2}} & \frac{\pm k_3}{2\sqrt{k^2+m_0^2}} & 0 & \frac{\pm m_0+1}{2\sqrt{k^2+m_0^2}} \end{pmatrix}. \tag{71}$$

We thus obtain for the negative and positive parts $\hat{H}_- = \hat{P}_-\hat{H} = \hat{H}\hat{P}_-$ and $\hat{H}_+ = \hat{P}_+\hat{H} = \hat{H}\hat{P}_+$ of $\hat{H}$ respectively:

$$\hat{H}_\pm = \left(\hat{H} \pm \sqrt{k^2+m_0^2}I_4\right)/2. \tag{72}$$

Hence

$$\hat{H}_+ - \hat{H}_- = \sqrt{k^2+m_0^2}I_4, \tag{73}$$

where I_4 denotes the 4×4 unit matrix. To translate $\hat{H}_\pm$ into a coordinate space representation, that is, to calculate $H_\pm$, one has simply to replace in $\hat{H}_\pm$ the k_j by $-\mathrm{i}\partial/\partial x_j$ and in particular $\sqrt{k^2+m_0^2}$ by $\sqrt{-\Delta+m_0^2}$, $\Delta = \sum_{j=1}^{3} \partial^2/\partial x_j^2$. This yields

$$H_\pm = \left(H \pm \sqrt{-\Delta+m_0^2}I_4\right)/2 \tag{74}$$

and thus

$$H_- - H_+ = \sqrt{-\Delta+m_0^2}I_4 \tag{75}$$

(note that the operators $H_\pm$ have absolutely continuous spectra and are mutually orthogonal).

By a diagonalization of $\hat{H}$ it would be possible to decompose the 4-spinor field into the direct sum of eigenvectors for the negative and positive spectral parts respectively (cf [13]). However for calculations to be performed here this would be of no use. Another way to circumvent the indefiniteness of H is (cf [14]) to introduce a new Hilbert space $\mathcal{H}_j$ by defining a new inner product

$$\langle f, g\rangle_J := \mathrm{Re}\langle f, g\rangle + \mathrm{i}\,\mathrm{Im}((P_+ - P_-)f, g) = \langle \hat{f}, \hat{g}\rangle_J = \mathrm{Re}\langle \hat{f}, \hat{g}\rangle + \mathrm{i}\,\mathrm{Im}((\hat{P}_+ - \hat{P}_-)\hat{f}, \hat{g}),$$

where $\hat{f}$ and $\hat{g}$ are the Fourier transforms of f and g respectively and P_+ and P_- are the correspondents of $\hat{P}_+$ and $\hat{P}_-$ respectively in coordinate space. As regards the Hamilton operator this boils down to replace H by $|H| \equiv H_- - H_+ = \sqrt{-\Delta+m_0^2}I_4$ when operating in $\mathcal{H}_j$, that is, $\langle f, H_J g\rangle_J \equiv \langle f, |H|g\rangle$. The only way to define in a canonical way a strictly contractive semi-group of maps $\tilde{\alpha}_t, t \geqslant 0$, is obviously to choose $\tilde{\delta}_{|H|} = |H| + |H|^\times$ as its generator. Following the scheme proposed in [14], that is, using the Hilbert space $\mathcal{H}_J$ (or rather $\mathbb{H} = \mathcal{H}_J \otimes \mathcal{H}_J^\times$) dynamics is then to be determined by the unitary group of maps α_t with a generator $\delta_{|H|} = \mathrm{i}(|H| - |H|^\times)$. Since $|H| = \sqrt{-\Delta+m_0^2}I_4$ this means that w.r.t. $\tilde{\alpha}_t = \exp[-t(|H| + |H|^\times)]I_4$ and $\alpha_t = \exp[\mathrm{i}t(|H| - |H|^\times)]I_4$ we have a complete decomposition of the spinor state space into a direct sum of its components. Hence we may assume that $\Lambda = \oplus_{i=1}^{4}\Lambda^{(i)}$ where each $\Lambda^{(i)}$ is a copy of the operator Λ calculated in section 6, meaning that in each $\mathbb{H}^{(i)}$ the temporal behaviour with regard to α_t and $\tilde{\alpha}_t$ would be as calculated in section 6. Note however that one may also assume an intertwining of components by choosing a different decomposition of Λ for the here considered spinor state space.

The above calculations can easily be extended so as to include a constant electromagnetic potential (A_0, A_1, A_2, A_3). Let $p^2 = \sum_{j=1}^{3}(k_j - eA_j)^2$. Then the eigenvalues of $\hat{H}$ are $\nu_1 = eA_0 - \sqrt{m_0^2 + p^2}$ and $\nu_2 = eA_0 + \sqrt{m_0^2 + p^2}$ each with multiplicity 2. Let $\hat{H}_1$ and $\hat{H}_2$ be the corresponding projections of $\hat{H}$ so that $\hat{H}_{21} = \hat{H}_2 - \hat{H}_1$ has eigenvalues $\mu_1 = -\nu_1 = -eA_0 + \sqrt{m_0^2 + p^2}$ and $\mu_2 = \nu_2 = eA_0 + \sqrt{m_0^2 + p^2}$. Let $|\mathbf{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$. If $0 \leqslant \pm A_0 \leqslant |\mathbf{A}|$ then $\hat{H}_{21} \geqslant 0$. If $|\mathbf{A}| \leqslant A_0$ then $\hat{H} \geqslant 0$ and if $|\mathbf{A}| \leqslant -A_0$ then $-\hat{H} \geqslant 0$. In any of these cases neither $\hat{H}_{21}$ nor $\hat{H}$ are diagonal if $A_0 \neq 0$. However if $A_0 = 0$ then $\hat{H}_{21} = \sqrt{m_0^2 + p^2} I_4$ so that in coordinate space $H_{21} = \sqrt{m_0^2 + (\mathrm{i}\nabla - e\mathbf{A})^2} I_4, \mathbf{A} = (A_1, A_2, A_3)$. Detailed calculations will be left to further investigations.

10. Intrinsic stochasticity of classical systems

In case of a quantum mechanical Hamilton operator $H \geqslant 0$ an intrinsic stochasticity could be defined in a canonical fashion via the contractive semi-group $\tilde{\alpha}$ of maps $\tilde{\alpha}_t : A \to \exp(-t\tilde{\delta}_H)A, \tilde{\delta}_H = H + H^\times, t \geqslant 0$. This construction fails for classical systems since the classical dynamical groups of time translations consist of outer automorphisms $\alpha_t = \exp(tL_h)$, where L_h is the Liouville operator of the corresponding Hamilton function h. There is however a possibility of imitating the above construction as follows. Let Ξ be a quantization map from an algebra $\mathcal{A}^{(\mathrm{cl})}$ of classical observables into an algebra $\mathcal{A}$ of quantum mechanical observables. Let $\alpha_t^{(\mathrm{cl})}$ and α_t be the dynamical automorphisms on $\mathcal{A}^{(\mathrm{cl})}$ and $\mathcal{A}$ respectively. Assume that Ξ is dynamically faithful, that is, $\alpha_t(\Xi a) = \Xi\big(\alpha_t^{(\mathrm{cl})}a\big)$ for all $t \in \mathbb{R}$ and $a \in \mathcal{A}^{(\mathrm{cl})}$. Define then a contractive semi-group of maps $\tilde{\alpha}_t^{(\mathrm{cl})}$ by $\tilde{\alpha}_t(\Xi a) = \Xi\big(\tilde{\alpha}_t^{(\mathrm{cl})}a\big), t \geqslant 0$. By some limiting process one obtains then $\tilde{\alpha}_t^{(\mathrm{cl})} = \exp(-2th(p, q))$, where $h(p, q)$ is the Hamilton function of the considered classical system (the same result could have been obtained by simply replacing in the operator expression $\mathrm{e}^{-tH} f \mathrm{e}^{-tH}$ the operator product by the ordinary commutative product and the operator H by the corresponding classical Hamilton function). Since the dynamical group of a classical (conservative) system with a Hamilton function $h(z), z = (p, q) \in \mathbb{R}^{2n}$, is generated by the Liouville operator, $L_{h(z)} = \sum_{j=1}^{n}[(\partial h(z)/\partial p_j)\partial q_j - (\partial h(z)/\partial q_j)\partial p_j]$, one obtains by differentiation of $\tilde{\alpha}_t\Lambda = \Lambda\alpha_t$ w.r.t. t followed by setting $t = 0$ and partial integration

$$\begin{aligned} 0 &= 2h(z)(\Lambda f)(z) + (\Lambda L_h f)(z) \\ &= \int_{\mathbb{R}^{2n}} [2h(z)\lambda(z, w) f(w) + \lambda(z, w) L_{h(w)} f(w)]\,\mathrm{d}w \\ &= \int_{\mathbb{R}^{2n}} [2h(z)\lambda(z, w) - (L_{h(w)}\lambda(z, w))] f(w)\,\mathrm{d}w, \\ w &= (u, v) \in \mathbb{R}^{2n}. \end{aligned}$$

Here we have assumed that

$$\lim_{|w_j|\to\infty} \lambda(z, w) f(w) = 0, \qquad 1 \leqslant j \leqslant n. \tag{76}$$

So $\lambda(z, w)$ must satisfy

$$2h(z)\lambda(z, w) - L_{h(w)}\lambda(z, w) = 0. \tag{77}$$

Setting $F(z, w) = \log(\lambda(z, w))$ we get

$$(L_{h(w)}F)(z, w) = 2h(z). \tag{78}$$

The problem is to find a particular solution for this linear partial differential equation of first order. For if $F_1(z, w)$ is such a solution then the general solution is (due to $(L_{h(w)}h)(w) = 0$)

$$F(z, w) = F_1(z, w) + F_0(z, h(w)), \tag{79}$$

where $F_0(z, h(w))$ is (up to certain mild differentiability conditions) arbitrary. As a simple easily solvable example we take $h(z) = m_0|p|^2/2$, $|p|^2 \equiv p_1^2 + \cdots + p_n^2$. Then

$$\lambda(z, w) = \exp\left[F_0(z, h(w)) + m_0|p|^2 \sum_{j=1}^{n} v_j/u_j\right]. \tag{80}$$

The temporal behaviour of a state function $f(p, q)$ is thus generally given by a Laplace transform

$$\left\|\tilde{\alpha}_t^{(\text{cl})}\Lambda f\right\|^2 = \int_{\mathbb{R}^{2n}} \exp(-4th(p, q))|(\Lambda f)(p, q)|^2 d(p, q). \tag{81}$$

Let in the above example $n = 1$ and choose $F_0(z, h(w)) = -m_0^2 p^4/(4u^2) - q^2/2 + \log(g(p))$, where $g(p)$ is positive and $|g(p)|\exp(-p^2) \to 0$ for $|p| \to \infty$. Let further $f(u, v) = \exp(-u^2 - v^2)$. If $g(p) = 1$ then

$$\left\|\tilde{\alpha}_t^{(\text{cl})}\Lambda f\right\|^2 = (\pi^3/2)/\sqrt{2(m_0 + t)}. \tag{82}$$

This extends easily to arbitrary n because $\lambda(z, w) = \Pi_{j=1}^n \lambda(z_j, w_j)$. That is, we obtain with $f(w) = \exp(-|u|^2 - |v|^2)$ the relation

$$\left\|\tilde{\alpha}_t^{(\text{cl})}\Lambda f\right\|^2 = (\pi^3/2)^n/(2(m_0 + t))^{n/2}. \tag{83}$$

(which obviously is valid only for $t > -m_0$) with an asymptotic expansion $\sim 1/\sqrt{t}$. Basically the same overall behaviour (in terms of Bessel functions) results from choosing $g(p) = \exp(-p^4)$. Choosing $g(p) = \exp(-p^{2n})$, $n \geqslant 3$, delivers an expression (in terms of hypergeometric functions) for $\left\|\tilde{\alpha}_t^{(\text{cl})}\Lambda f\right\|^2$ which is defined and strictly decreasing for all $|t| < \infty$ and an asymptotic expansion $\sim 1/\sqrt{t}$. That is, we get basically the same results as in the case of a nonrelativistic quantum mechanical free particle. It follows further that in all cases $\langle \mathbf{1}, \Lambda f\rangle = c_0\langle \mathbf{1}, f\rangle$ (with $c_0 > 0$ depending on the particular choice of the function $g(p)$) and that Λ preserves positivity.

Consider now a classical relativistic particle with rest-mass m_0. Its Hamiltonian is $h(p) = c\sqrt{c^2 m_0{}^2 + p^2}$. Hence

$$F(p, q; u, v) = 2h(p)h(u)v/(c^2 u) + F_0(p, q; u, v).$$

Let

$$F_0(p, q; u, v) = -2vh(p)h(u)/(c^2u^2) - h(p)[h(p)-)h(u)/c^2]/u^2 - q^2/2 + \log(\gamma(p)),$$

where $\gamma(p)$ is positive and uniformly bounded. Then for $f(u, v) = \exp(-u^2 - v^2)$ one gets after a short calculation

$$|(\tilde{\alpha}_t\Lambda f)(p, q)|^2 = \pi\gamma(p)^2 \exp[-4(t + c)h(p)/c]\exp(-q^2).$$

Choosing $\gamma(p) = \sqrt{|p|}$ and integrating w.r.t. p and q yields

$$\|\tilde{\alpha}_t\Lambda f\|^2 = \pi^{3/2}\exp[-4cm_0(c + t)][1 + 4cm_0(c + t)]/[4(c + t)]^2 \tag{84}$$

which holds for $t > -c$. For $\gamma(p) = \sqrt{|p|}\exp[-h(p)^2/c^2]$ one obtains

$$\begin{aligned}\|\tilde{\alpha}_t\Lambda f\|^2 = {}& \exp[-c^2 m_0(4 + m_0)]\exp[-4cm_0(c + t)] \\ & \times \{1/2 + \sqrt{\pi}(c + t)\exp[(cm_0 + 2(c + t))^2] \\ & \times [-1 + \Phi(cm_0 + 2(c + t))]\},\end{aligned} \tag{85}$$

where Φ is the error function. The asymptotic behaviour is given by

$$\begin{aligned}\|\tilde{\alpha}_t\Lambda f\|^2 = {}& \exp[-c^2 m_0(4 + m_0)]\exp[-4cm_0(c + t)] \\ & \times \{1/2 + \sqrt{\pi}(c + t)\exp[(cm_0 + 2(c + t))^2]\} \\ & \times \{1 - 2(c + t)[(cm_0 + 2(c + t))^{-1} \pm O((cm_0 + 2(c + t))^{-2})]\}.\end{aligned} \tag{86}$$

This agrees qualitatively with the result for the quantum mechanical free Hamiltonian. The function $t \to \|\tilde{\alpha}_t \Lambda f\|^2$ defined by (85) is strictly decreasing and defined for all $t \in \mathbb{R}$. (Calculations for a three-dimensional spherically symmetric Hamiltonian of a free particle are somewhat more tedious, the result is however similar to the one-dimensional case considered above.)

To outline a general scheme we start with the following example. Let $h(u,v) = u^2/2 + V(v)$, $(u,v) \in \mathbb{R}^2$, $V \in C^1(\mathbb{R})$. Set $x = h(u,v)$, $y = v$ and $F(p,q;u,v) = g(p,q;x,y)$. Then

$$(L_{h(u,v)}F)(p,q;u,v) = u\partial_y g(p,q;x,y).$$

Now, $u = \sqrt{2(x - V(y))}$. Thus equation (78), reading now $u\partial_y g(p,q;x,y) = 2h(p,q)$, can be integrated yielding

$$F(p,q;u,v) = 2h(p,q)\int \mathrm{d}y/\sqrt{2(x - V(y))}\Big|_{x=h(u,v),\,y=v}. \tag{87}$$

As an example choose $V(v) = \alpha/v^2$, $\alpha > 0$. Then the above formula delivers

$$F(p,q;u,v) = uvh(p,q)/h(u,v). \tag{88}$$

As a second example choose $h(\pi_\rho, \pi_\varphi, \rho) = \left(\pi_\rho^2 + \pi_\varphi^2/\rho\right)/2 + V(\rho)$, that is a radially symmetric Hamiltonian with two degrees of freedom. Set $\xi_1 = h(\pi_\rho, \pi_\varphi, \rho)$, $\xi_2 = \pi_\varphi$, $\eta = \rho$ and $F(p_r, p_\varphi, r; \pi_\rho, \pi_\varphi, \rho) = g(\pi_\rho, \pi_\varphi, \rho; \xi_1, \xi_2, \eta)$. Then $\left(L_{h(\pi_\rho,\pi_\varphi,\rho)}F\right)(p_r, p_\varphi, r; \pi_\rho, \pi_\varphi, \rho) = \pi_\rho \partial_\eta g(\pi_\rho, \pi_\varphi, \rho; \xi_1, \xi_2, \eta)$. Substituting $\pi_\rho = \sqrt{2\xi_1 - \xi_2^2/\eta^2 - 2V(\eta)}$ one gets

$$F(p_r, p_\varphi, r; \pi_\rho, \pi_\varphi, \rho) = 2h(p_r, p_\varphi, r)\int \mathrm{d}\eta\Big/\sqrt{2\xi_1 - \xi_2^2/\eta^2 - 2V(\eta)}\Big|_{\xi_1=h(\pi_\rho,\pi_\varphi,\rho),\,\xi_2=\pi_\varphi,\,\eta=\rho}. \tag{89}$$

As a final example we choose the Hamiltonian of a two-dimensional Toda chain,

$$h(u,v) = |u|^2/2 + \mathrm{e}^{v_1-v_2} + \mathrm{e}^{v_2-v_1}, \qquad |u|^2 = u_1^2 + u_2^2, \qquad u = (u1, u_2), \qquad v = (v_1, v_2).$$

Set $k_1 = h(u,v)$, $k_2 = u_1 + u_2$, $k_3 = -\mathrm{e}^{v_1-v_2} + \mathrm{e}^{v_2-v_1} + u_1u_2$, $l = \mathrm{e}^{v_1-v_2} + \mathrm{e}^{v_2-v_1}$ and $F(p,q;u,v) = g(p,q;k_1,k_2,k_3,l)$, $(p,q) = (p_1,p_2,q_1,q_2)$. Then

$$(L_{h(u,v)}F)(p,q;u,v) = (\mathrm{e}^{v_1-v_2} - \mathrm{e}^{v_2-v_1})(u_1 - u_2)\partial_l g(p,q;k_1,k_2,k_3,l). \tag{90}$$

A short calculation yields

$$u_1 - u_2 = \pm 8\sqrt{4k_1 - k_2^2 - 4l}, \mathrm{e}^{v_1-v_2} - \mathrm{e}^{v_2-v_1} = \pm\sqrt{l^2 - 4}.$$

Thus (note that k_3 is already determined by l)

$$F(p,q;u,v) = \pm(h(p,q)/8)\int \mathrm{d}l\Big/\sqrt{(l^2-4)\left(4k_1 - k_2^2 - 4l\right)}\Big|_{k_1=h(u,v),\,k_2=u_1+u_2,\,l=\mathrm{e}^{v_1-v_2}+\mathrm{e}^{v_2-v_1}} \tag{91}$$

(the integral yields an elliptic function).

Summing up it is easily seen that all examples above are integrable systems in which we had used their first integrals $x, x_1, x_2, \xi_1, \xi_2, k_1, k_2, k_3$ as auxiliary variables in order to integrate equation (78). (The integration variables y, η, l could have been chosen as arbitrary functions of the remaining phase space variables. However, for convenience we had chosen in each example the most simple function.)

To construct an operator K satisfying $\alpha_t K = K\tilde{\alpha}_t$ (this includes the case $K = \Lambda^{-1}$) we copy the procedure just used to calculate an integral kernel λ for Λ. Let $\kappa(z,w)$ be an integral kernel for K. Proceeding now as in the calculation of λ we arrive at

$$L_{h(z)}G(z,w) = -2h(w), \qquad \kappa(z,w) = \exp[G(z,w) + G_0(h(z),w)], \tag{92}$$

where $G_0(z, w)$ satisfies $Lh(z)G_0(z, w) = 0, z = (p, q), w = (u, v)$. As an example consider $h(z) = m_0 p^2/2$. Then $G(z, w) = -m_0 u^2 q/p + G_0(p, u, v)$.

Remark 9. Relation (78) can be simplified by setting $F(z, w) = 2h(z)s(w)$, where $s(w)$ is a solution of $(\sharp)L_{h(w)}s(w) = 1$. Thus $s(w)$ is the 'phase function' (or 'time function') of the system determined by $h(w)$ (cf [18]). Writing $E \equiv h(p, q)$ relation ($\sharp$) reads $\{E, s\}_{\text{Poisson}} = 1$. That is, $\{E, s\}$ is a canonical couple that can be used to replace another couple of conjugate variables. The name 'phase function' refers to the fact that for conservative systems the time dependence of a solution for coordinate and momentum functions is determined up to a 'phase'. Thus $s(w)$ can be obtained by eliminating the time variable t from the dynamical equations (meaning that one has actually to solve the dynamical equations). If for example $h(p, q) = p^2/2 + V(q), (p, q) \in \mathbb{R}^2$, then $s(p, q)(\equiv t(p, q)) = \int^q \mathrm{d}x/\sqrt{2E - 2V(x)}$, and for $h(p_r, p_\varphi, r) = p_r^2/2 + p_\varphi/r^2 + V(r)$ we get $s(p_r, p_\varphi, r) = (\equiv t(p_r, p_\varphi, r)) = \int^r \mathrm{d}x/\sqrt{2E - 2V(x) - p_\varphi^2/x^2}$ (which is exactly what we had obtained above).

11. Reverse processes, Λ- and *K*-measurements

For a time-flow $\mathbb{R} \ni t \to \alpha_t$ and a semi-group of contracting maps $\tilde{\alpha}_t$ we generally define now an operator K by

$$\alpha_t K = K\tilde{\alpha}_t \tag{93}$$

(note that this includes the case $K = \Lambda^{-1}$). It follows then from the definitions of Λ and K

Proposition 23.

$$K\Lambda = K\tilde{\alpha}_{-t}\tilde{\alpha}_t\Lambda = \alpha_{-t}K\Lambda\alpha_t, \tag{94}$$

$$\Lambda K = \Lambda\alpha_t\alpha_{-t}K = \tilde{\alpha}_t\Lambda K\tilde{\alpha}_{-t}. \tag{95}$$

Because of these relations we shall say that K defines a *reverse process*. Differentiation w.r.t. t followed by setting $t = 0$ one gets

Corollary 15.

$$[H, K\Lambda] = 0, \tag{96}$$

$$[H, \Lambda K] = 0, \tag{97}$$

where both commutator relations are to be understood to hold on their respective domains. That is, both $K\Lambda$ and ΛK as well as their adjoints and $^\times$-transforms are constants of motion. (Note that neither $K\Lambda$ nor ΛK are necessarily multiples of the identity.)

As an example for the construction of an integral kernel for K let us consider a free nonrelativistic Hamiltonian in one dimension (the multi-dimensional case being an easy generalization). That is, $H = P^2, H^\times = P^{\times 2}, \alpha_t = \exp[\mathrm{i}t(H - H^\times)]$ and $\tilde{\alpha}_t = \exp[-t(H + H^\times)]$ (a factor $\hbar/2m_0$ is implicit in a scaling $t \to t\hbar/2m_0$). Using Fourier transformation and regularization one obtains for an integral kernel κ of K,

$$\kappa(x, y, u, v) = \int_{\mathbb{R}^2} \hat{\gamma}(k, l)\exp\{\mathrm{i}[k(\sqrt{\mathrm{i}}x - u) + l(\sqrt{-\mathrm{i}}y - v)]\}\,\mathrm{d}(k, l), \tag{98}$$

where $\hat{\gamma}(k, l)$ is arbitrary up to conditions similar to those set up for the function $\hat{g}(k, l)$ in section 5. For $K = \Lambda^{-1}$ the condition is $\hat{\gamma}(k, l) = 1/\hat{g}(k, l)$. K is densely defined on $C_c(\mathbb{R}^2)$

This can be proved by showing that the characteristic function of any finite rectangle in $\mathbb{R}^2$ is mapped by K into an unbounded and non-integrable element of $\left(L^1_{\text{loc}} \cap C\right)(\mathbb{R}^2)$ whereas Λ maps $C_c(\mathbb{R}^2)$ into $\mathcal{S}(\mathbb{R}^2)$ (this holds correspondingly for n-dimensions). We note without proof that for $\bar{\tilde{\gamma}}(k,l) = \hat{\gamma}(l,k)$ there holds $K = K^\times$ and that K is volume preserving, that is, $\langle \mathbf{1}, Kf\rangle = c_f\langle \mathbf{1}, f\rangle$, $c_f = \text{const} \geqslant 0$ for all $f \in C_c(\mathbb{R}^2) \cup \mathcal{S}(\mathbb{R}^2)$. Let $\kappa\lambda$ and $\lambda\kappa$ denote the integral kernels of $K\Lambda$ and Λ respectively. Then for $\hat{g}(k,l) = \hat{\gamma}(k,l) = \exp(-k^2 - l^2)$ one gets

$$\kappa\lambda(x, y, u, v) = 2\pi^3 \exp[-\mathrm{i}(x + y - u - v)(x - y - u + v)/4], \tag{99}$$

$$\lambda\kappa(x, y, u, v) = 2\pi^3 \exp\{-[(x - u)^2 + (y - v)^2]/8\}. \tag{100}$$

It follows from these relations that $K\Lambda$ and ΛK map $\mathcal{S}(\mathbb{R}^2)$ into itself. By a Λ- or K-*measurement* we mean a map

$$A \to \langle \Lambda\alpha_t f, A\Lambda\alpha_t f\rangle = \langle \tilde{\alpha}_t \Lambda f, A\tilde{\alpha}_t \Lambda f\rangle \\ \equiv \tau(\tilde{\rho}_{\Lambda,t} A), \tilde{\rho}_{\Lambda,t} = (\tilde{\alpha}_t \Lambda f)(\tilde{\alpha}_t \Lambda f)^* \tag{101}$$

or

$$A \to \langle \alpha_t K f, A\alpha_t K f\rangle = \langle K\tilde{\alpha}_t f, AK\tilde{\alpha}_t f\rangle \\ \equiv \tau(\tilde{\rho}_{K,t} A), \tilde{\rho}_{K,t} = (K\tilde{\alpha}_t f)(K\tilde{\alpha}_t f)^* \tag{102}$$

respectively.

Corollary 16.

$$A\alpha_t = \alpha_t A \Rightarrow \partial_t \langle K\tilde{\alpha}_t f, AK\tilde{\alpha}_t f\rangle = 0. \tag{103}$$

We give an example for a Λ-measurement. Let $H = P^2$ and (cf section 5) $\hat{g}(k,l) = \exp(-k^2 - l^2)$. Choose now $A = A^\times = \beta_1(P^2 + P^{\times 2}) + \beta_2(Q^2 + Q^{\times 2})$, $\beta_1, \beta_2 = $ real constant. Let further $f_n(x, y) := x^n y^n \exp(-x^2 - y^2)$, $n = 0, 1, 2, \ldots$. Then some lengthy calculations yield

$$\langle \Lambda\alpha_t f_0, A\Lambda\alpha_t f_0\rangle \equiv F_0(t, \beta_1, \beta_2) = \pi^5[4\beta_1 + 17\beta_2 + 16t(2 + t)\beta_2]/4(t + 1)^2,$$

$$\langle \Lambda\alpha_t f_{2m}, A\Lambda\alpha_t f_{2m}\rangle \equiv F_{2m}(t, \beta_1, \beta_2) = \pi^5 2^{4m-2}[(2m - 1)!!]^2(1 + t)^{2m-2} \\ \times [4\beta_1 + (17 + 2m)\beta_2 + 16t(2 + t)\beta_2]/[17 + 16t(2 + t))]^m, \qquad m \geqslant 1,$$

$$\langle \Lambda\alpha_t f_{2m+1}, A\Lambda\alpha_t f_{2m+1}\rangle = 0.$$

It follows from these relations:

$$\lim_{t\to\pm\infty} F_0(t, \beta_1, \beta_2) = \lim_{t\to\pm\infty} F_2(t, \beta_1\beta_2) = 4\pi^5\beta_2,$$

$$\lim_{t\to\pm\infty} F_{2m}(t, \beta_1, \beta_2) = 4\pi^5\beta_2[(2m - 1)!!]^2, \qquad m > 1,$$

$$\lim_{t\to -1} F_0(t, \beta_1, \beta_2) = \infty,$$

$$\lim_{t\to -1} F_2(t, \beta_1\beta_2) = 4\pi^5(4\beta_1 + 3\beta_2),$$

$$\lim_{t\to -1} \partial_t^k F_{2m}(t, \beta_1, \beta_2) = 0, \qquad 2 < m, \quad 0 \leqslant k \leqslant 2m - 1.$$

The $F_{2m}(t, \beta_1, \beta_2)$ are positive and symmetric w.r.t. $t = -1$ and with the exception of $F_0(t, \beta_1\beta_2)$ (which has a pole of second order at $t = -1$) are defined on the whole real axis. $F_2(t, \beta_1, \beta_2)$ has maximum at $t = -1$, all the other functions $F_{2m}(t, \beta_1, \beta_2)$ have one local minimum at $t = -1$ and two maxima at

$$t = -1 \pm (1/4)[(m - 1)(4\beta_1 + \beta_2 + 2m\beta_2)/(4\beta_1 + \beta_2 + m\beta_2)]^{-1/2}.$$

Note that the functions $F_{2m}(t,1,0)$, $m \geqslant 0$, that is, $A = P^2 + P^{\times 2} \equiv H + H^{\times}$, have basically the same characteristics as the cases where $\beta_2 \neq 0$.

We state without proof that for $m \geqslant 1$

$$\lim_{T\to\infty}(1/T)\int_{-1}^{T} F_{2m}(t,\beta_1,\beta_2)\,\mathrm{d}t = \lim_{t\to\infty} F_{2m}(t,\beta_1,\beta_2)$$

from which it follows by the symmetry of $F_{2m}(t,\beta_1,\beta_2)$ w.r.t. $t = -1$ that

$$\lim_{T\to\infty}(1/T)\int_{-T}^{T} F_{2m}(t,\beta_1,\beta_2)\,\mathrm{d}t = 2\lim_{t\to\infty} F_{2m}(t,\beta_1,\beta_2).$$

Remark 10. Letting $A = A^{\times} = \beta_1(P + P^{\times})^2 + \beta_2(Q + Q^{\times})^2$ yields the same expressions for $F_{2m}(t,\beta_1,\beta_2)$. However

$$F_1(t,\beta_1,\beta_2) = \pi^5\beta_2/(t+1),$$
$$F_{2m+1}(t,\beta_1,\beta_2) = \pi^5\beta_2(1+t)^{2m-1}2^{4m}[(2m+1)!!]^2/[17+16t(2+t)]^m, \qquad m > 0.$$

That is, $\lim_{t\to\pm\infty} F_{2m+1}(t,\beta_1,\beta_2) = 0$, $m \geqslant 0$, and the $F_{2m+1}(t,\beta_1,\beta_2)$ are skew-symmetric w.r.t. $t = -1$. $F_1(t,\beta_1,\beta_2)$ has a pole of first order at $t = -1$ whereas the $F_{2m+1}(t,\beta_1,\beta_2)$ are bounded and defined on the whole real axis.

A K-measurement for $A = \beta_1(P^2 + P^{\times 2}) + \beta_2(Q^2 + Q^{\times 2}$ is not possible because $\langle\alpha_t Kf, A\alpha_t Kf\rangle$ does not exist for any $f \in C_c(\mathbb{R}^2)$. A measurement is however possible for the operator $B = \exp(-Q^2 - Q^{\times 2})$ yielding (with f_n as above) $\langle\alpha_t Kf_n, B\alpha_t Kf_n\rangle = 0$ if n is odd and

$$\langle\alpha_t Kf_n, B\alpha_t Kf_n\rangle = 2^4\pi^5[(n-1)!!]^2(19+36t+16t^2)^n/(5+4t)^{2+2n}$$

if n is even. The corresponding Λ-measurement yields

$$\langle\Lambda\alpha_t f_n, B\Lambda\alpha_t f_n\rangle = \pi^5 2^{4n}[(n-2)!!]^2$$
$$\times (1+4t)^{2n-2}(37+48t+16t^2)^{(n-1)/2}/[(5+4t)^{2n}(17+32t+16t^2)^{(n-1)/2}]$$

if n is even and $\langle\Lambda\alpha_t f_n, B\Lambda\alpha_t f_n\rangle = 0$ if n is odd. In both measurements the (only) singularity is at $t = -5/4$. Compared with the Λ-measurements for the operator A which had with one exception no singularities this is somewhat unexpected since A is an unbounded operator whereas B is a bounded operator with a purely continuous spectrum whose null space contains only the zero function.

Remark 11. Let A be any operator (including the identity) and denote by Π_t either $\langle\Lambda\alpha_f, A\Lambda\alpha_f\rangle$ or $\langle\alpha_t Kf, A\alpha_t Kf\rangle$. Depending on the specific properties of Λ or K as well as the generator of α_t it can happen, as the above examples show, that (up to a finite number of values for t) Π_t exists for $-\infty \leqslant t_0 \leqslant t$. That is, there could simultaneously exist processes in which $\mathrm{d}\Pi_t/\mathrm{d}t$ might be negative as well as positive (see also section 8), with relations (96) and (97) suggesting a balance between both processes. Interpreting for example for particles a Λ-process as a decay, a K-process could be interpreted as the generation of a particle. But, as already remarked in the introduction, these interpretations remain so far hypothetical.

12. Concluding remarks

Though the subject treated here goes a long way back to papers by Prigogine *et al* as well as [5], we think that it has not yet been thoroughly exploited. For this reason we have taken it up again, trying to demonstrate that there is a number of features worth to be treated more exhaustively. Using computer programs which now allow comparatively sophisticated formal calculations we have been able to deliver some results which had been hitherto inobtainable on a pedestrian way. We think that these results might justify further investigations. This should include in more detail stochastic processes of the types described in sections 3 and 4 as well as the spontaneous decay of particles, in particular half-spin type ones, for which the calculations in sections 5, 6, 7 and 9 might serve as a rough illustration, further coupled and 'large' systems both classical and quantum mechanical, that is, systems in which problems connected with asymptotic stability and euilibrium viz nonequilibrium play a dominant role. It is obvious that calculations for such examples are considerably more difficult than the examples treated here.

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Appendix A.

The statements (S1) to (S5) below had been proved in [5].

(S1) The function $t \to \|M^{1/2} f\|^2 - \|M^{1/2}\alpha_t f\|^2$, $f \in D_M$, $t \geqslant 0$, is positive nondecreasing and has a limit for $t \to \infty$.

(S2) If $\lim_{t\to\infty} \|\Gamma^{1/2}\alpha_t f\|$, $f \in D_M$, exists, then it is zero; in this case $\lim_{t\to\infty}\langle g, \Gamma\alpha_t f\rangle = 0$ for all $g \in \mathrm{dom}(\Gamma^{1/2})$.

(S3) Let $\alpha = \{\alpha_t = \exp(\mathrm{i}t\delta) | t \in \mathbb{R}\} \subset \mathcal{L}(\mathbb{H})$ be a strongly continuous unitary group, let E_δ denote the spectral measure of δ, and let $\mathbb{H}_a(\delta)$ denote the set of $f \in \mathbb{H}$ for which the function spectrum $(\delta) \to \mathbb{R}_+ : \mu \to \|E(\mu) f\|^2$ is absolutely continuous. Let finally $G(t) \equiv \int_0^1 \alpha^*{}_{ts}\Gamma\alpha_{ts}\,\mathrm{d}s(\text{strongly})$, $t \geqslant 0$. Then

$$\mathbb{H}_a \supset \mathrm{ran}((M + \mu\mathbf{1})^{-1} t G(t)(M + \mu\mathbf{1})^{-1}, \qquad \mu > 0.$$

If in particular 0 is not in the point spectrum of Γ then δ is absolutely continuous.

(S4) If $0 \notin \{\langle f, \Gamma f\rangle | f \in \mathrm{dom}(\Gamma), \|f\| = 1\}$ then M has an empty point spectrum.

(S5) If $M = \Lambda^*\Lambda + k\mathbf{1}$, $k \geqslant 0$, is bounded then $\alpha_g^* M\alpha_g = M$ for all $g \in \mathcal{G}_+$.

Let $\mathcal{D}_\mathbb{H}$ be a dense linear subspace of $\mathbb{H}$ and let $\mathcal{D}'_\mathbb{H}$ denote the linear space of all $g : \mathcal{D}_\mathbb{H} \to \mathbb{C} : g \to g(f)$ such that $g(f) := \langle f, g\rangle$ is finite. (In case $\mathbb{H}$ is a space of $\mathbb{C}$-valued function one can identify $\mathcal{D}_\mathbb{H} \subset \mathbb{H} \subset \mathcal{D}'_\mathbb{H}$ with a Gel'fand triple.) If T is a linear map with domain and range in $\mathbb{H}$, let $\overline{\mathrm{dom}}(T) := \{g \in \mathcal{D}'_\mathbb{H} | T(g) \in \mathcal{D}'_\mathbb{H}\}$.

(S3) can be extended as follows: (S3 i) *Let* $\mathrm{i}\delta := \mathrm{s} - \lim_{t\to 0}\alpha_t$ *and assume that* $\delta(g) = 0$, $g \in \mathcal{D}'_\mathbb{H}$, *and* $M^{-1}g \in \overline{\mathrm{dom}}(M)$. *Then* $\Gamma(M^{-1}g) = 0$. *If* $\Gamma(f) = 0, 0 \neq f \in \mathcal{D}'_\mathbb{H}$, *then* $\alpha_t(Mf) = Mf$ *for all* $t \in \mathbb{R}$.

Proof. $\delta(g) = 0$ implies $\alpha_t(Mf) = Mf$, $f = M^{-1}g$, for all $t \in \mathbb{R}$. Thus

$$0 = d(\alpha_t g)/\mathrm{d}t = (\mathrm{d}(\alpha_t^* M\alpha_t\alpha_t^* M^{-1}g)/\mathrm{d}t = -\alpha_t^*\Gamma\alpha_t\alpha_t^*(M^{-1}g) = -\alpha_t^*\Gamma(M^{-1}g),$$

hence $\Gamma(M^{-1}g) = 0$. If $\Gamma f = 0$ then

$$0 = \alpha_t^*\Gamma\alpha_t\alpha_t^* f = -d(\alpha_t^* M\alpha_t\alpha_t^* f)/\mathrm{d}t = \mathrm{d}(\alpha_t^* Mf)/\mathrm{d}t.$$

Hence $\alpha_t^*(Mf) = Mf$ and thus $Mf = \alpha_t(Mf)$. □

Appendix B.

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A} = \mathcal{L}(\mathcal{H})$, and let $\bar{\mathcal{A}}$ be the set of (not necessarily bounded) linear operators with domain and range in $\mathcal{H}$. Let further $\mathbb{H} = L^2(\mathcal{A}, \text{trace})$, $\mathcal{A}^{(1)} = \mathcal{L}(\mathbb{H})$, and let $\bar{\mathcal{A}}^{(1)}$ be the set of (not necessarily bounded) linear operators with domain and range in $\mathbb{H}$. Recall that $T^\times f := (Tf^*)^*$. $\bar{\mathcal{A}}$ is a subset of $\bar{\mathcal{A}}^{(1)}$ in the following sense: if $A \in \bar{\mathcal{A}}$ then the maps $T_A : f \to Af$ and $T_A^\times : f \to A^\times f = fA^*$, $f, f^* \in \text{domain}(A) \subset \mathbb{H}$, are in $\bar{\mathcal{A}}^{(1)}$. Clearly, the set of maps $\{T_A, T_A^\times | A \in \bar{\mathcal{A}}\}$ does not cover $\bar{\mathcal{A}}^{(1)}$. Consider for example an infinite set of momentum operators $p_n \in \bar{\mathcal{A}}^{(1)}$, $n \in \mathbb{N}$, (let for example $p_n = U_n^* p U_n$, $n \in \mathbb{N}$, where $U_n : \mathbb{H} \to \mathbb{H}$ is unitary). Let (cf section 7) $P_n = a_n p_n + b_n p_n^\times$, $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$. Taking an average over the P_n^2, say in the strong limit, one has

$$P^{(2)} := s - \lim_{N\to\infty} (1/N) \sum_{n=1}^{N} P_n^2 \equiv P_a^{(2,0)} + P_b^{(0,2)} + P_{(a,b)}^{(1,1)}$$

where

$$P_a^{(2,0)} = s - \lim_{N\to\infty} (1/N) \sum_{n=1}^{N} a_n^2 p_n^2,$$

$$P_b^{(0,2)} = s - \lim_{N\to\infty} (1/N) \sum_{n=1}^{N} b_n^2 p_n^{\times 2}$$

and

$$P_{a,b}^{(1,1)} = s - \lim_{N\to\infty} (2/N) \sum_{n=1}^{N} a_n b_n p_n p_n^\times .$$

Now, $P_a^{(2,0)}$ and $P_b^{(0,2)}$ are elements of $\bar{\mathcal{A}}$ (considering $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}^\times = \{A^\times | A \in \mathcal{A}\}$ as identical). But this is not true for $P_{a,b}^{(1,1)}$ which is an element of $\bar{\mathcal{A}}^{(1)}$ (but not of $\bar{\mathcal{A}}$). Since the construction of $P^{(2)}$ resembles closely the case of a variable in statistical thermodynamics which has to be determined by an infinite number of (an average over infinitely many) dynamical variables, we call elements $T = \sum_{n=1}^{\infty} A_n B_n^\times$, where $A_1, B_1, A_2, B_2, \ldots \in \bar{\mathcal{A}}$, *thermodynamic variables*. Denote by $\mathcal{L}_{\mathcal{A}}(\mathbb{H})$ the set of those maps T in which the $A_1, B_1, A_2, B_2, \ldots$ are in $\mathcal{A}$. It was shown in [5] that $\mathbb{H}$ is isomorphic $\mathcal{H} \otimes \mathcal{H}^*$. Let

$$U(t) = u(t_1)u(t_2)^\times, \qquad V(t) = v(t_1)v(t_2)^\times, \qquad t = (t_1, t_2) \in \mathbb{R}^2,$$

where (u, v) is a Weyl couple, that is,

$$u(x)v(y) = \exp(\mathrm{i}xy)v(y)u(x), \qquad x \in \mathbb{R}, \quad y \in \mathbb{R}.$$

Then (U, V), too, is a Weyl couple for which holds

$$U(t)V(s) = \exp(\mathrm{i}st)V(s)U(t), \qquad st \equiv s_1t_1 + s_2t_2.$$

That is,

$$U(t)V(s)f = u(t_1)v(s_1)f v(s_2)u(t_2), \qquad f \in \mathbb{H},$$

is equivalent to writing

$$(u(t_1)v(s_1) \otimes v(s_2)u(t_2))(f_1 \otimes f_2) = (u(t_1)v(s_1)f_1) \otimes (v(s_2)u(t_2)f_2), \qquad f \simeq f_1 \otimes f_2.$$

Now, the algebra generated by the Weyl couple (U, V) is a subset of $\mathcal{L}_{\mathcal{A}}(\mathbb{H})$, and its strong closure and consequently the strong closure of $\mathcal{L}_{\mathcal{A}}(\mathbb{H})$ yields $\mathcal{L}(\mathbb{H})$. Hence the operators Λ, M, Γ (due to properties proved above) are in general thermodynamic variables.

Remark 12. In some cases it can be convenient to use a Weyl representation when solving equation (1), say for its generators, that is,

$$(H + H^{\times})\Lambda + \mathrm{i}\Lambda(H - H^{\times}) = 0.$$

Appendix C.

Recall that the step $\mathcal{H} \to \mathbb{H} = L^2(\mathcal{A}, \tau)$, $\mathcal{A} \to \mathcal{A}^{(1)} \equiv \mathcal{L}(\mathbb{H})$ was necessary by the fact that the operator Λ was shown to be an element of $\bar{\mathcal{A}}^{(1)}\backslash\bar{\mathcal{A}}$ (that is, cannot be an element of $\bar{\mathcal{A}}$). This situation might repeat itself (due to Lyapunov viz Λ-processes of higher complexity) thus giving rise to an ascending chain of algebras, $\mathcal{A} \subset \mathcal{A}^{(1)} \subset \cdots \subset \mathcal{A}^{(k)} \subset \cdots$ with state spaces $\mathbb{H}^{(k)} \equiv L^2(\mathcal{A}^{(k)}, \tau^{(k)}) \simeq \mathbb{H}^{(k-1)} \otimes \mathbb{H}^{(k-1)}$, $\mathcal{A}^{(k)} \equiv \mathcal{L}(\mathbb{H}^{(k-1)})$, $\mathbb{H}^{(0)} \equiv \mathbb{H}$, $\mathcal{A}^{(0)} \equiv \mathcal{A}$. Such algebras and spaces will also be necessary when considering for example systems with particles with different rest-masses. This can be achieved w.r.t. the above provided composite state model in the following way. Let $a(p)$ and $b(p) = a^{\times}(p)$ be as defined in this model of a (linear) field with rest-mass m_0. Let $A(p)$ the creation operator of a field with rest-mass M and define

$$A(p) = \nu(p)[\hat{\rho}(p)a(p) + \hat{\sigma}(p)b(-p)],$$

where $\nu(p) = \left(\mu_M(p)/\mu_{m_0}(p)\right)^{1/2}$ and $\hat{\rho}$ and $\hat{\sigma}$ are smooth real- or complex-valued functions which are arbitrary up to the condition $\nu(p)^2[\hat{\rho}(p)^2 + \hat{\sigma}(p)^2] = 1$. The composite states generated by $A(p)$ are the eigenstates of the Hamiltonian

$$\mathbf{H} = \int_{\mathbb{R}^3} \mu_M(p)A^*(p)A(p)\,\mathrm{d}(p).$$

The field operator of the compound system (mass M) is $\Phi = \rho * \varphi + \sigma * \varphi^{\times}$, where ρ and σ are the inverse Fourier transforms of $\hat{\rho}$ and $\hat{\sigma}$, respectively, φ and $\varphi^{\times}$ are the particle and antiparticle field operators, respectively, and $*$ means convolution. The momentum operator corresponding to Φ is $\Pi = \kappa * \pi + \lambda * \pi^{\times}$, where κ and λ are the inverse Fourier transforms of $2\mu_M\hat{\rho}$ and $2\mu_M\hat{\sigma}$, respectively, and π and $\pi^{\times}$ are the momentum operators of the field φ and φ^*, respectively.

In the next step we assume $a(p)$ and $a(p)^*$ to be creation and annihilation operators of a compound system and decompose them similarly as was done with $A(p)$ and $A(p)^*$. That is,

$$a(p) = \nu_1(p)[\hat{\rho}_1(p)a_1(p) + \hat{\sigma}_1(p)b_1(-p)],$$

where $\nu_1(p) = \left(\mu_{m_0}(p)/\mu_{m_1}(p)\right)^{1/2}$, $\nu_1(p)^2[\hat{\rho}_1(p)^2 + \hat{\sigma}_1(p)^2] = 1$, $b_1(p) = a_1^C(p)$ and $a_1 \to a_1^C$ is an involution which corresponds to the involution $a \to a^{\times}$ in the preceding step. The states created by the $A(p)$ are then sums of four-fold tensor products

$$\begin{aligned} A(p) &= \beta_1 a_1(p) + \beta_2 a_1^{\times}(-p) + \beta_3 a_1^{\times C}(-p) + \beta_4 a_1^{\times C}(p) \\ &= \beta_1(a_1 \otimes 1 \otimes 1 \otimes 1)(p) + \beta_2(1 \otimes a_1 \otimes 1 \otimes 1)(-p) \\ &\quad + \beta_3((1 \otimes 1 \otimes a_1 \otimes 1)(-p) + \beta_4(1 \otimes 1 \otimes 1 \otimes a_1)(p). \end{aligned}$$

That is, $A(p)$ operates on $\mathbb{H}^{(1)} \otimes \mathbb{H}^{(1)} = \otimes_{j=1}^4 \mathbb{H}$. Subsequent steps are, if necessary, obvious from this scheme.

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